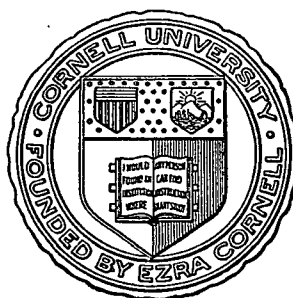# CORNELL UNIVERSITY

GRADUATE SCHOOL OF AEROSPACE ENGINEERING

FOCUSING OF FINITE-AMPLITUDE CYLINDRICAL AND SPHERICAL

SOUND WAVES IN A VISCOUS AND HEAT-CONDUCTING MEDIUM

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BIOGRAPHICAL SKETCH

The author was born in ▐�▐▐▐▐▐▐▐ on ▐▐▐▐▐▐▐. He was graduated with honors from Cheng Kung University, Taiwan in June, 1964, where he received the B. S. degree in Mechanical Engineering.

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IN MEMORY OF MY LATE FATHER

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ABSTRACT

The focusing of spherical and cylindrical pulses in a viscous and heat-conducting medium has been investigated. It is found that this problem can be studied efficiently by dividing the region of interest into three regions, namely, the converging region, the interaction region, and the diverging region.

The convergence of cylindrical and spherical pulses of small amplitude in a viscous and thermally conducting medium is governed by a linear equation with a viscous and heat-conduction term. The solution of this equation shows that the amplitude of the pulse decreases initially because of viscous diffusion and heat conduction, then increases rapidly due to the strong convergence near the origin. Any initial discontinuity contained in the pulse profile is diffused to a transition zone whose width becomes the order of the pulse-length as the pulse approaches the interaction region. In the interaction region, the small-amplitude pulse is governed to first order by the inviscid linear wave equation. A Fourier transform technique is used to solve the equation; the solution thus obtained shows that the amplitude of the focusing pulse remains small in the whole interaction region including the axis or center of symmetry. In the spherical case, the peculiar phenomenon of 180-degree phase change of the pulse-form after it is reflected from the center of symmetry is found.

The convergence of cylindrical and spherical pulses of finite amplitude in a medium of very small viscosity and heat conductivity is governed

by the radial Burgers equation with a small parameter ε^* multiplying the term with second derivatives. The method of matched asymptotic expansions is found applicable to this equation; a composite solution obtained describes a converging pulse with changing pulse-form, e.g., a sinusoidal wave distorts to an N-wave. The flow field inside the shocks contained in the pulse (e.g., the front or rear shock of the N-wave) is locally described by Taylor's shock structure. In the interaction region, no small-perturbation solution exists for the shocks under the conditions $\frac{\varepsilon}{\delta} = 0(1)$, $\delta \ll 1$, and $\delta R_e \gg 1$. However, the flow field between the shocks (e.g., the flow field between the front and rear shocks of an N-wave) satisfies to first order the inviscid linear wave equation, which can also be solved by the same Fourier transform technique. This approximation is justified only if the parameters ε , δ , R_e satisfy certain relations which insure that the amount of energy dissipated due to the entropy rise across a shock is negligible in comparison with the total energy of the pulse.

The divergence of the reflected pulses is briefly studied. The treatment of the diverging pulse is similar to that of the converging pulse except for trivial changes in the analysis.

INTRODUCTION

The propagation of cylindrically and spherically symmetric waves has been discussed extensively in the literature. Diverging symmetrical waves have received much attention because of their connection with a whole series of phenomena, e.g., bomb explosions, electrical discharges, thunder, sonic booms, balloon bursting, etc. Converging symmetrical waves have been studied in relation to the problems of focusing of ultrasound, cavitation, and the intensification of sonic booms.

Lamb[1] and Jeffreys and Jeffreys[2] applied inviscid linear acoustic theory to study the focusing of waves resulting from an idealized explosion such as the bursting of a spherical balloon. The mathematical model used is that a pressure-jump discontinuity is located at a spherical boundary which separates two regions of different uniform pressure. The prssure inside the spherical boundary is slightly higher than the outer pressure. At an initial time, the boundary is broken and motion is allowed to begin; an expansion wave propagates toward the center of the sphere and a compression wave propagates outward into the quiescent fluid. The linear inviscid solution to the initial-value problem shows that the expansion wave (a jump discontinuity) increases its strength as it converges and becomes inifinite strength as it arrives at center of symmetry. However, Lamb found that the instantaneous infinite strength at the center could be avoided if the initial pressure-jump discontinuity at the spherical boundary was replaced by a smooth but rapid pressure transition across the spherical boundary. In this way the converging expansion wave

possesses an artificial structure which causes pressure to remain finite at the center as the wave arrives. The physical situation near the center, as a converging expansion wave arrives there, is described by Resler[3], who notes that the absolute pressure behind the wavefront will reach zero (vacuum) and the velocity of fluid particles approaches a limiting value U_{max} ; both are finite at center.

Lamb's idea of avoiding the infinite strength at center of converging spherical expansion wave can be achieved naturally if viscous effects are considered. The study of plane waves in a viscous medium by Lagerstorm et al.[4], Hanin[5], and Knudson[6] reveals that the viscous effects modify the jump discontinuity into a rapid but smooth transition. Dunn and Dove[7] have examined the focusing of spherical waves resulting from an idealized explosion in a viscous medium using the linear equations with viscous term. As expected, the linear viscous solution closely resembles Lamb and Jeffreys' inviscid one, the major differences being that the step and delta functions in the inviscid solution are replaced by a modified error function and a Gaussian function, respectively, and the strength of converging expansion wave remains finite at the center of symmetry.

The convergence of a weak cylindrical compression wave with a discontinuous wavefront, i.e., a shock discontinuity, has been treated by Zababakhin and Nechaev[8] in a linear inviscid approximation. They found that the strength of the compression wave increases without bound and the pressure on the front of the reflected wave has logarithmic singular behavior. Later these results were confirmed by Zel'dovich[9] using a

different method. The convergence of a weak spherical shock was examined in linear inviscid approximation by Chernous'ko[10]. He obtained a similarity solution to the linear wave equation with spherical symmetry, which shows that the intensity of the converging wave becomes infinite as the shock wave reaches the center, and the phenomenon of phase change after the reflection at center as indicated by the similarity solution: a compression wave reflects as an expansion wave (and conversely), after which the undisturbed state is re-established. The above investigations about converging cylindrical and spherical waves of discontinuous front in a uniform medium illustrate the fact that the linear inviscid solutions are singular at the axis or center of symmetry. A possible way to improve the singular behavior at origin of the linear solution is to use the nonlinear theory, since the strength of the wave becomes large in the vicinity of the axis or center of symmetry. The motion of a strong converging cylindrical or spherical shock wave in a inviscid perfect gas with constant specific heats has been studied by Guderley[11], Stanyukovich[12], and Sedov[13]. The basic equations of fuild dynamics without the dissipation terms are used. The strong converging shock is assumed moving into a uniform medium initially at rest. As the shock converges, it becomes very strong; the strong-shock conditions are applicable: the pressure in front of the shock is neglected in comparison with the pressure behind the shock and the density behind the strong shock becomes constant. The use of strong-shock conditions leads to a boundary condition for the flow behind the shock wave which permits a similarity

solution to the problem, including a description of the resulting outward-
going shock. This similarity solution shows, the strenghtening of the
converging shock and singular behavior at the axis or center of sym-
metry. Chisnell[14] treated the same problem with a different approach
and obtained similar result. The singular behavior at the axis or center
of symmetry noted in the linearized description still persists in the
nonlinear similar solution; although it changed qualitatively.

In an actual flow situation, the occurrence of unbounded velocity,
pressure etc., is unrealistic, because the real-gas effects such as
viscosity, thermal conductivity , radiation etc., might significantly
modify the results of the nondissipative gas flow theory. The numerical
calculation of a converging cylindrical shock by Payne[15], in which an
artificial diffusion term is induced in the finite-difference equation
as the differential equations are approximated by the difference equations,
shows that the solution remains finite when the shock arrives the axis.
Dtachenko and Limshennik[16] solved numerically the problem of converging
cylindrical shock in a fully ionized plasma taking account the dissipa-
tive effects; the result also shows bounded behavior at the axis. These
numerical calculations demonstrate the importance of the dissipative
effects in the shock-focusing problem. Chapman[17] examined analytically
the problem of a weak converging cylindrical shock in a viscous and heat
conducting medium. Later, Manickam[18] extended Chapman's results to
the weak converging spherical shock. Their results show that for a range
of "d", which is the ratio of the shock strength to the square root of
a reciprocal Reynolds number, a weak inward-travelling shock wave in a

viscous, heat-conducting fluid remains of small strength in the whole

flow, including the axis and center of symmetry. However, their method

yields the solution in the plane of subcharacteristic coordinates, which

itself depends on the solution; the mapping of the solution back into

the physical plane requires the numerical solution of an ordinary non-

linear differential equation.

The aim of the present study is, in the context of small-perturba-

tion theory[19], to investigate the converging cylindrical and spherical

pulses and consequent reflected pulses in a viscous and thermally con-

ducting medium. A pulse is a disturbance that does not affect a given

fluid parcel until some definite instant and than lasts only for a finite

time interval; the change of entropy of fluid parcel will persist but is

3rd-order locally. For example, a propagating N-wave is a pulse. It is

well known that the cylindrical and spherical pulses are different from

a plane pulse in one respect; as the cylindrical or spherical pulse passes

through a given point, both compressions ($P - P_0 > 0$) and rarefactions

($P - P_0 < 0$) will always be observed at that point, while in the plane

case an individual compression pulse or rarefaction pulse may exist. Thus,

the pulses considered in this study contain both compressions and rare-

factions. The change of pulse-form as well as the focusing phenomenon

at the axis and center of symmetry of the converging pulse are the

main co   ms in this study.

Ir   apter II, we formulate the mathematical model for the converg-

ing pulse by stating the basic equations and accompanying initial and

boundary conditions. Three fundamental assumptions made in the analysis

are: (1) All gas variables in the region of space under study differ but slightly from the corresponding values in the undisturbed state; (2) the characteristic length of the pulse (pulse-length) is small compared to the initial radius R_0, whence the pulse starts to propagate; (3) the viscosity and thermal conductivity are very small. We divide the flow field of our converging-pulse problem into two regions, namely, the converging region and interaction region. The converging region is defined as the region between the initial position of the pulse R_0 and the boundary of the interaction region. The interaction region is defined as the region whose radius is the order of the pulse-length and it includes the axis or center of symmetry.

In Chapter III we examine the pulse in the converging region. The second-order equation for this region is found to be Burgers' equation with an additional geometrical term. It is known that Burgers' equation[20] provides a good approximation for the study of finite-amplitude waves in a viscous medium in the plane case. A general solution of Burgers' equation was obtained independently by Hopf[21] and Cole[22]. But the presence of the additional geometrical term in our equation destroys the applicability of the Hopf-Cole transformation. However, the method of matched asymptotic expansions[23], [24] is found to be capable of dealing with the approximate equation, provided the conditions $\frac{\varepsilon}{\delta} = 0(1)$, $R_e \varepsilon \gg 1$ are fulfilled. Here ε is the ratio of the gas particle velocity to the undisturbed speed of sound; δ is the ratio of the pulse-length to the initial position of the pulse R_0 ; R_e is the Reynolds number based on pulse-length and the undisturbed gas variables.

The case of linear acoustics in viscous medium is also studied from the
approximate equation by assuming the condition $\frac{\varepsilon}{\delta} \ll 1$.

In Chapter IV, we investigate the focusing phenomenon of the converging pulse in the interaction region. The existence of both incoming and reflected outgoing waves in this region indicates that the approximate equation should have a solution consisting of waves of both families. For a pulse of infinitesimal amplitude, the approximate equation for this region is a linear wave equation with radial symmetry, which is solved by the method of Fourier transforms. The asymptotic solution at large distance away from the origin and the limiting solution near the origin are obtained and discussed. For the case of weak nonlinear pulse, i.e., $\frac{\varepsilon}{\delta} = 0(1)$, $R_e \varepsilon \gg 1$, shocks develop as the pulse converges. Using an N-wave as our example, we investigate the front shock and the flow field between the front and rear shocks. It is found that the portion of pulse where the shocks are located always violates the assumption of small perturbation as $r \to 0$ even though nonlinear and viscous effects are accounted for. The portion of pulse excluding the shocks is shown to satisfy the linear wave equation to first order. This equation is used to study the focusing of an N-wave in the interaction region.

Finally, we discuss briefly the diverging pulse in Chapter V. The treatment of the diverging pulse is identical to that of the converging pulse treated in Chapter III except for trivial changes in the analysis.

II. FORMULATION

We now formulate our mathematical model of the convergence of a cylindrical or spherical pulse. An inward propagating cylindrical or spherical pulse of pulse-length λ is presumed to be located initially in a cylindrical or spherical shell between $r = R_0 + \dfrac{\lambda}{2}$ and $r = R - \dfrac{\lambda}{2}$, outside which the medium is at rest or nearly so. We wish to know the motion of the inward propagating pulse at the subsequent times.

The basic equations for this problem are the equations for one-dimensional flow, for which all gas variables depend on time t and on one single geometrical coordinate r , which determines the distance from the axis or center of symmetry. These are the continuity, momentum, energy equations, equation of state, and the perfect-gas law:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} = -j \frac{\rho u}{r} \tag{1}$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = -\frac{\partial p}{\partial r} + \frac{4}{3} \mu \left[\frac{\partial^2 u}{\partial r^2} + j \frac{1}{r} \frac{\partial u}{\partial r} - j \frac{u}{r^2} \right] \tag{2}$$

$$\rho T \left(\frac{\partial S}{\partial t} - u \frac{\partial S}{\partial r} \right) = \left[\frac{\partial^2 T}{\partial r^2} + \frac{j}{r} \frac{\partial T}{\partial r} \right] + \frac{4}{3} \mu \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{u}{r} \right)^2 - j \frac{u}{r} \frac{\partial u}{\partial r} \right] \tag{3}$$

$$\frac{p}{p_0} = \left[\frac{\rho}{\rho_0} \right]^{\gamma} e^{(S - S_0)/C_v} \tag{4}$$

$$p = \rho R T \tag{5}$$

Here u , ρ , p , T and S are the gas variables, respectively
the velocity of gas particles, density, pressure, temperature, and
specific entropy. The parameters occurring in these equations are: the
space parameter j , which is equal to 1 for cylindrically symmetric
flows and 2 for spherically symmetric flows; the coefficients of vis-
cosity μ and of heat conductivity k , both of which are assumed
constants; the gas constant R ; the ratio of specific heats $\gamma = \dfrac{C_p}{C_v}$;
and the specific heat at constant volume C_v . The suffix zero here
refers to values of the dependent variables at some undisturbed standard
state. The coefficient of bulk viscosity μ_B of the gas is assumed to
be zero. The calculations can be extended to the case of non-zero bulk
viscosity by modifying the factor $\dfrac{4}{3}$ in Eqs.(2) & (3) to the factor
$(\dfrac{4}{3} + \dfrac{\mu_B}{\mu})$.

Nondimensionalization of the Basic Equations

We would like to nondimensionalize the system of equations (1) to
(5) in accordance with our converging-pulse problem. The dimensionless
variables, which are denoted by primed quantities, are defined by the
following relations:

$$u' = \frac{u}{a_o}$$

$$\rho' = \frac{\rho - \rho_o}{\rho_o}$$

$$p' = \frac{p - p_o}{p_o} \tag{6}$$

$$T' = \frac{T - T_0}{T_0}$$

$$S' = \frac{S - S_0}{C_v}$$

along with the dimensionless independent variables

$$r' = \frac{r}{R_0}$$

$$t' = \frac{a_0 t}{R_0} \tag{7}$$

Here R_0, a_0, p_0, ρ_0, T_0 and S_0 are respectively the initial position of the central point of the pulse, the undisturbed sound speed, pressure, density, temperature and specific entropy.

If relations (6) and (7) are substituted in the basic equations (1) to (5) and the primes are dropped, the nondimensional equations are

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + (1 + \rho) \frac{\partial u}{\partial r} = -j (1 + \rho) \frac{u}{r} \tag{8}$$

$$(1 + \rho) \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = -\frac{1}{\gamma} \frac{\partial p}{\partial r} + \frac{1}{Re_{R_0}} \left[\frac{\partial^2 u}{\partial r^2} + \frac{j}{r} \frac{\partial u}{\partial r} - j \frac{u}{r^2} \right] \tag{9}$$

$$(1 + \rho)(1 + T) \left(\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial r} \right) = \frac{K}{Re_{R_0}} \left[\frac{\partial^2 T}{\partial r^2} + \frac{j}{r} \frac{\partial T}{\partial r} \right]$$

$$+ \frac{\gamma(\gamma - 1)}{Re_{R_0}} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{u}{r} \right)^2 - j \frac{u}{r} \frac{\partial u}{\partial r} \right] \tag{10}$$

$$(1 + P) = (1 + \rho)(1 + T) \tag{11}$$

$$(1 + P) = e^{S}(1 + \rho)^{\gamma} \tag{12}$$

The nondimensional parameter $Re_{R_o} = \frac{3}{4}\frac{a_o \rho_o R_o}{\mu}$ is $\frac{3}{4}$ of the Reynolds number based on R_o & a_o , and the nondimensional parameter $K = \frac{3}{4}\frac{k}{\mu C_v}$ $= \frac{3\gamma}{4}\frac{1}{Pr}$, where $P_r = \frac{\mu C_p}{k}$ is the Prandtl number. This system of equations (8) to (12) is nonlinear and no exact solution is available. In such situations, the usual technique is to make simplifying assumptions which will be appropriate to some physical problem. Under some assumptions the system of equations can be properly reduced to a relatively simple forms which may be solved analytically or numerically.

Fundamental Assumptions

In our converging pulse problem we make the following assumptions on the initial disturbances and the nature of the medium:

(i) The initial disturbances of the gas variables (the primed quantities in (6)) differ slightly from the corresponding values in the undisturbed state (the quantities with suffix zero in (6)); they are all of the same order, i.e., $u' \sim p' \sim \rho' \sim T' \sim S' \sim 0(\varepsilon)$. Where the nondimensional parameter ε is defined as

$$\varepsilon = \frac{P - P_o}{P_o} \ll 1 \tag{13}$$

The disturbances shall constitute a single wave propagating inward.

(ii) The characteristic length of the pulse λ (pulse-length) is small in comparison to the initial radius of the central point of the pulse R_0 . Thus, we have another small nondimensional parameter δ , which is defined as

$$\delta = \frac{\lambda}{R_0} \ll 1 \tag{14}$$

(iii) The viscosity and thermal conductivity are such that the Reynolds number based on pulse-length λ , Re_λ , is very much larger than unity,

$$Re_\lambda = \frac{a_0 \rho_0 \lambda}{\mu} \gg 1 \tag{15}$$

It is important to note that the second assumption provides a natural way to study the converging pulse. From these assumptions, there can be only one family waves at a time, i.e., either incoming waves or outgoing waves in the region defined by $\delta \ll r \leq 1$, but waves of both families, i.e., incoming waves and reflected outgoing waves, may exist and interact in the region defined by $r \leq \delta$. The region defined by $\delta \ll r \leq 1$ is called converging region and the region defined by $r \leq \delta$ is called interaction region. Since the size of the converging region is much greater than the length of the converging pulse, the propagation of the converging pulse in this region is considered a long-time process. While the size of the interaction region is comparable to the pulse-length, the pulse stays in this region only for a short time. We will study each region separately; however, the

solutions of these two regions describe whole flow field for the converging pulse.

Initial and Boundary Conditions

The initial condition imposed on the system of equations (8) to (12) is that an inward-propagating cylindrical or spherical pulse of given profile is initially located in a thin shell region between $r = 1 + \frac{\delta}{2}$ and $r = 1 - \frac{\delta}{2}$, where r is the nondimensional space variable. The boundary condition is that there is no mass flux into or out of the origin $r = 0$; the required solution should have no motion at the axis or center of symmetry for all time.

III. CONVERGING REGION

In this chapter we study the converging pulse. The basic equations are the system of equations (8) to (12), from which we will derive the approximate equation. The main simplification comes from the fact that only one family of waves at a time needs to be considered in this region. In order to study the converging pulse the following coordinates are introduced:

$$\eta = r$$

$$\xi = \frac{t + r}{\delta} \tag{16}$$

where $\delta = \dfrac{\lambda}{R_o} \ll 1$ is defined by (14).

The fundamental assumption of small perturbation of all gas variables (Eq.(13)) allows us to expand the gas variables in powers of a small nondimensional parameter ε :

$$u(\eta, \xi; \varepsilon) = \varepsilon \, u_1(\eta, \xi) + \varepsilon^2 u_2(\eta, \xi) + \cdots$$

$$p(\eta, \xi; \varepsilon) = \varepsilon \, p_1(\eta, \xi) + \varepsilon^2 p_2(\eta, \xi) + \cdots$$

$$\rho(\eta, \xi; \varepsilon) = \varepsilon \, \rho_1(\eta, \xi) + \varepsilon^2 \rho_2(\eta, \xi) + \cdots$$

$$S(\eta, \xi; \varepsilon) = \varepsilon \, S_1(\eta, \xi) + \varepsilon^2 S_2(\eta, \xi) + \cdots$$

$$T(\eta, \xi; \varepsilon) = \varepsilon \, T_1(\eta, \xi) + \varepsilon^2 T_2(\eta, \xi) + \cdots$$

$$\tag{17}$$

Substitution of relations (16) and (17) into the system of equations (8) to (12) yields a sequence of approximate equations; the first-order and second-order equations are

$O(\frac{\varepsilon}{\delta}, 1)$:

$$\frac{\partial \rho_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} = 0$$

$$\frac{\partial u_1}{\partial \xi} + \frac{j}{\gamma} \frac{\partial P_1}{\partial \xi} = 0$$

$$\frac{\partial S_1}{\partial \xi} = 0$$

$$P_1 - \rho_1 = T_1$$

$$P_1 - \gamma \rho_1 = S_1$$

$$\tag{18}$$

$O(\frac{\varepsilon^2}{\delta}, \varepsilon, \frac{\varepsilon}{Re_{R_0} \delta^2})$:

$$\frac{\varepsilon}{\delta}\left(\frac{\partial \rho_2}{\partial \xi} + u_1\frac{\partial \rho_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} + \rho_1\frac{\partial u_1}{\partial \xi}\right) + \frac{\partial u_1}{\partial \eta} + j\frac{u_1}{\eta} = 0$$

$$\frac{\varepsilon}{\delta}\left(u_1\frac{\partial u_1}{\partial \xi} + \frac{1}{\gamma}\frac{\partial P_2}{\partial \xi} + \rho_1\frac{\partial u_1}{\partial \xi} + \frac{\partial u_2}{\partial \eta}\right) + \frac{1}{\gamma}\frac{\partial P_1}{\partial \eta} - \frac{1}{Re_{R_0}\delta^2}\frac{\partial^2 u_1}{\partial \xi^2} = 0$$

$$\frac{\varepsilon}{\delta}\left(\frac{\partial S_2}{\partial \xi} + \rho_1\frac{\partial S_1}{\partial \xi} + T_1\frac{\partial S_1}{\partial \xi} + u_1\frac{\partial S_1}{\partial \xi}\right) + \frac{K}{Re_{R_0}\delta^2}\frac{\partial^2 T_1}{\partial \xi^2} = 0$$

$$P_2 - \rho_2 - T_2 - \rho_1 T_1 = 0$$

$$P_2 - \gamma \rho_2 - T_2 - \frac{\gamma(\gamma-1)}{2}\rho_1^2 = 0$$

$$\tag{19}$$

Upon integrating the equations (18) with respect to ξ and using the fact that the disturbances vanish at $\xi = -\infty$, we get the relations

$$u_1 = -\rho_1 = -\frac{p_1}{\gamma}$$

$$S_1 = 0$$

$$T_1 = (\gamma-1)\rho_1 \tag{20}$$

The relations (20) are used to simplify the second-order equations (19), from which a single equation for u_1 is finally obtained.

$$\frac{\partial u_1}{\partial \eta} + \frac{j}{2}\frac{u_1}{\eta} - \frac{\varepsilon\Gamma}{\delta}u_1\frac{\partial u_1}{\partial \xi} = -\frac{1}{Re\,\delta}\frac{\partial^2 u_1}{\partial \xi^2} \tag{21}$$

where

$$\frac{1}{Re} = \frac{1}{Re_{R_0}\,\delta}\left(\frac{1+K(\gamma-1)}{2}\right) = \frac{1}{Re_\lambda}\left(\frac{1+K(\gamma-1)}{2}\right)$$

$$\Gamma = \frac{\gamma+1}{2} \tag{22}$$

Equation (21) governs the motions of the converging pulse in the converging region, in which the nolinear, viscous and heat conducting effects are taken into account. It is important to note that Eq.(21) is derived from a second-order calculation, as may be seen from (19), but only the first-order quantity u_1 appears in the equation. This situation is often encountered in perturbation calculations, both with ordinary and partial differential equations. In our problem the non-linear, viscous, and heat-conducting effects are indeed small quantities. The "second-order" terms embodying the quantities will, over long periods of time, cause a first-order distortion of the motion considerably. Since the propagation of the pulse in the converging region

involves long times, i.e., the pulse travels a distance much greater than its own pulse-length, the nonlinear, viscous, and heat-conducting effects must be included in the approximate equation as they make a first-order contribution. Equation (21) is the Burgers equation with an additional geometrical term $\frac{j}{2} \frac{u_1}{\eta}$; it may be called the radial Burgers equation. It is known that Burgers' equation provides a good approximation for the study of finite-amplitude waves with viscosity in plane geometry. A general solution of Burgers' equation has been found independently by Hopf[21] and Cole[22]. However, the Hopf-Cole transformation is not applicable to the radial Burgers equation (21) due to the presence of geometrical term $\frac{j}{2} \frac{u_1}{\eta}$; no general solution has yet been found. However, under certain conditions Eq.(21) can be solved analytically. In what follows we will study analytically two particular simplifications of Eq.(21), namely, the linear case in which $\frac{\varepsilon \Gamma}{\delta} \ll 1$, $\delta R_e = 0(1)$ and the weak nonlinear case in which $\frac{\varepsilon \Gamma}{\delta} = 0(1)$, and $\delta R_e \gg 1$. The condition $\frac{\varepsilon \Gamma}{\delta} = 0(1)$ is essential in analyzing the nonlinear effects on the propagation of pulses over a large distance.

Linear Case

In this case the amplitude of the disturbances characterized by ε is assumed so small in comparison with δ that the nonlinear term in Eq.(21), $\frac{\varepsilon \Gamma}{\delta} u_1 \frac{\partial u_1}{\partial \xi}$, can be dropped. Equation (21) becomes

$$\frac{\partial u_1}{\partial \eta} + \frac{j}{2} \frac{u_1}{\eta} = - \frac{1}{R_e \delta} \frac{\partial^2 u_1}{\partial \xi^2} \qquad (23)$$

Equation (23) governs the motions of converging pulse of infinitesimal amplitude in the converging region, taking the viscous and heat-conducting effects into account. The initial condition for Eq.(23) is that an inward propagating pulse of known profile is located in a thin shell between $r = 1 + \frac{\delta}{2}$ and $r = 1 - \frac{\delta}{2}$. This initial condition may be expressed in η , ξ coordinates as

$$u_1(\xi, \eta) = f(\xi) \qquad\qquad \text{at} \qquad \eta = 1 \tag{24}$$

The initial pulse-form $f(\xi)$ can be quite arbitrary; with u_1 a function of ξ alone only pulses moving inward are allowed.

To solve Eq.(23) with initial condition (24), we first introduct the transformations

$$u^* = u_1 \eta^{1/2}$$

$$\eta^* = -\eta$$

$$\xi^* = \sqrt{\delta Re}\, \xi \tag{25}$$

Upon substitution of the transformations (25) into Eq.(23) and initial condition (24), we have

$$\frac{\partial u^*}{\partial \eta^*} = \frac{\partial^2 u^*}{\partial \xi^{*2}} \tag{26}$$

$$u^*(\eta^*, \xi^*) = f\left(\frac{\xi^*}{\sqrt{\delta Re}}\right) \qquad \text{at} \quad \eta^* = -1 \tag{27}$$

Thus, the transformations have reduced Eq.(23) to Eq.(26), which is the well known one-dimensional heat-conduction equation. The solution of Eq.(26) with condition (27) can be easily found to be

$$u^*(\eta^*,\xi^*) = \frac{1}{2\sqrt{\pi(\eta^*+1)}} \int_{-\infty}^{\infty} f\left(\frac{\alpha}{\sqrt{\delta R_e}}\right) e^{-\frac{\left(\frac{\alpha}{\sqrt{\delta R_e}}-\xi^*\right)^2}{4(\eta^*+1)}} d\left(\frac{\alpha}{\sqrt{\delta R_e}}\right)$$

(28)

Rewriting above expression in terms of original variables u_1, η, ξ, we obtain the solution for Eq.(23) with initial condition (24)

$$u_1(\eta,\xi) = \frac{1}{2\eta^{1/2}} \frac{1}{\sqrt{\pi(1-\eta)}} \int_{-\infty}^{\infty} f\left(\frac{\alpha}{\sqrt{\delta R_e}}\right) e^{-\frac{\left(\frac{\alpha}{\sqrt{\delta R_e}}-\sqrt{\delta R_e}\xi\right)^2}{4(1-\eta)}} d\left(\frac{\alpha}{\sqrt{\delta R_e}}\right)$$

(29)

To illustrate the above solution more specifically, we select two initial pulse-forms as examples

$$f(\xi) = \xi e^{-\xi^2} \qquad \text{at} \quad \eta = 1 \qquad (30)$$

$$f(\xi) = \begin{cases} \xi & |\xi| \leq 1 \\ 0 & |\xi| > 1 \end{cases} \qquad \text{at} \quad \eta = 1 \qquad (31)$$

The function $f(\xi)$ in (30) is chosen because it is smooth for all ξ

and has a close resemblance to linear acoustic pulses actually produced in the laboratories. The function $f(\zeta)$ in (31) represents an N-wave with discontinuities at two fronts, $\zeta = \pm 1$. The N-wave is chosen because we wish to know how the viscosity and heat conductivity will affect this pulse-form, especially the discontinuities.

The solution of Eq.(23) with initial condition (30) is obtained by substituting (30) into (29) and evaluating the resultant integral; the result is

$$u_1(\eta, \zeta) = \frac{1}{\eta^{3/2}} \frac{1}{\left(1 + \frac{4(1-\eta)}{\delta Re}\right)^{3/2}} \zeta e^{-\frac{\zeta^2}{1 + \frac{4(1-\eta)}{\delta Re}}} \tag{32}$$

Solution (32) shows that the amplitude of the pulse is proportional to $\frac{1}{\eta^{3/2}} \cdot \frac{1}{\left(1 + \frac{4(1-\eta)}{\delta Re}\right)^{3/2}}$ and the pulse-length is characterized by the term $\left(1 + \frac{4(1-\eta)}{\delta Re}\right)$ in the exponential function. The amplitude is influenced by two contrary factors, the geometrical focusing and viscous heat-conducting effects represented respectively by $\frac{1}{\eta^{3/2}}$ and $\frac{1}{\left(1 + \frac{4(1-\eta)}{\delta Re}\right)^{3/2}}$. It is interesting to note that the amplitude decreases initially because of diffusion caused by viscosity and heat conduction and reaches its minimum value at certain η_m ; it increases rapidly due to strong focusing as $\eta \to 0$. The value η_m can be found by setting the derivative of the amplitude to zero:

$$\frac{d}{d\eta}\left[\frac{1}{\eta^{3/2}(1+\frac{4(1-\eta)}{\delta Re})^{3/2}}\right] = \frac{6}{\delta Re}\frac{1}{1+\frac{4(1-\eta)}{\delta Re}} - \frac{j}{2}\frac{1}{\eta} = 0$$

$$\therefore \quad \eta_m = \frac{4+\delta Re}{3j+4} \tag{33}$$

For high Reynolds number such that $\delta Re \geq 3j$, we obtain from (33) $\eta_m \geq 1$; thus the geometrical focusing dominates the whole flow field. For low Reynolds number, $\delta Re \to 0$, the viscous diffusion dominates in the region $1 \geq \eta > \frac{4}{3j+4}$. It is to be noted that the scale of pulse-length increases from 1 to $(1+\frac{4}{\delta Re})^{1/2}$ as $\eta \to 0$; thus for moderate Reynolds numbers, i.e., $\delta Re = 0(1)$, the pulse-length remains the same order of magnitude as the pulse approaches the interaction region.

The solution of Eq.(23) with initial condition (31) is obtained by substituting (31)into (29) and evaluating the resultant integral; the result is

$$u_1(\eta,\xi) = \frac{1}{\eta^{1/2}\sqrt{\delta Re}}\left\{\frac{\sqrt{1-\eta}}{\sqrt{\pi}}\left(e^{-\delta Re\left(\frac{\xi+1}{2\sqrt{1-\eta}}\right)^2} - e^{-\delta Re\left(\frac{\xi-1}{2\sqrt{1-\eta}}\right)^2}\right)\right.$$

$$\left.+ \sqrt{\delta Re}\,\xi\left[\Phi\left(\frac{\sqrt{\delta Re}\,(\xi+1)}{\sqrt{2}\,(1-\eta)}\right) - \Phi\left(\frac{\sqrt{\delta Re}\,(\xi-1)}{\sqrt{2}\,(1-\eta)}\right)\right]\right\}$$

$$\tag{34}$$

where

$$\Phi(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} e^{-\frac{u^2}{2}} du$$ is called the normal distribution function with

the special properties $\Phi(\infty) = 1$, $\Phi(0) = \frac{1}{2}$, $\Phi(-\infty) = 0$.

The important result from above solution is that the initial discontinu-

ities at $\xi = \pm 1$ are diffused by viscosity and heat conduction to two

transition zone with width of $O(\frac{\sqrt{1-\eta}}{\sqrt{\delta Re}})$) as the pulse converges (η

decreases). For moderate and low Reynolds number, i.e., $\delta Re \lesssim 1$, the

width of the transition zones becomes $O(1)$ as $\eta \rightarrow 0$; the initial

N-wave becomes a very smooth pulse-form as the pulse approaches the inter-

action region.

Weak Nonlinear Case

If the initial amplitude of the disturbances characterized by ε

is comparable to δ , i.e., $\frac{\varepsilon\Gamma}{\delta} = O(1)$, the nonlinear term in

Eq.(21) is as important as other terms and must be retained; the complete

equation (21) must be considered. We rewrite Eq.(21) here for convenience

$$\frac{\partial u_1}{\partial \eta} - \frac{j}{2}\frac{u_1}{\eta} - \frac{\varepsilon\Gamma}{\delta} u_1 \frac{\partial u_1}{\partial \xi} = -\frac{1}{Re\,\delta} \frac{\partial^2 u_1}{\partial \xi^2} \qquad (21)$$

The coefficient $\frac{\varepsilon\Gamma}{\delta}$ can be absorbed by using the new independent vari-

able ξ^* defined as

$$\xi^* = \frac{\delta}{\varepsilon\Gamma} \xi$$

$$(35)$$

In terms of the new variable ξ^* , Eq.(21) then becomes

$$\frac{\partial u_1}{\partial \eta} + \frac{j}{2} \frac{u_1}{\eta} - u_1 \frac{\partial u_1}{\partial \zeta^*} = - \varepsilon^* \frac{\partial^2 u_1}{\partial \zeta^{*2}} \qquad (36)$$

where

$$\varepsilon^* = \frac{\delta}{R_e \, \varepsilon^2 \Gamma} \doteq \frac{\mu}{\rho_0 \, a_0 \, R_0 \, \varepsilon^2 \Gamma} \qquad (37)$$

From the conditions imposed on the weak nonlinear case, i.e., $\frac{\varepsilon \Gamma}{\delta} = 0(1)$, $\delta R_e \gg 1$, we find that the new parameter $\varepsilon^* \ll 1$. Typically, for an acoustic pulse with amplitude $\varepsilon = 10^{-3}$ initially located at $R = 10$ ft and converging in air at standard condition, the parameter ε^* is $\approx (10^{-2})$. The initial condition is that an inward-propagating cylindrical or spherical pulse of given pulse-form occupies a cylindrical or spherical thin shell between $\eta = 1 + \frac{\delta}{2}$ and $\eta = 1 - \frac{\delta}{2}$. This initial condition may be expressed in η , ζ^* coordinates as follows

$$u_1(\eta, \zeta^*) = f(\zeta^*) \qquad \text{at} \qquad \eta = 1 \qquad (38)$$

Here $f(\zeta^*)$ is a given function which represents the pulse-form.

We should recognize that Eq.(36) with initial condition (38) constitutes a singular-perturbation problem with $\varepsilon^* \to 0$. A singular-perturbation problem[22], [23] arises when a second- or higher-order differential equation becomes a first- or lower-order one in the limit of a small parameter characterizing the problem approaching zero, i.e., in the limit $\varepsilon^* \to 0$. The solution of the limit equation of lower order, which is called outer solution, will not in general satisfy all the conditions imposed on the original equation. The regions where the outer

solution is not valid are called "boundary layers" . The approximate
equations for the boundary layers, which are called boundary-layer equa-
tions, are derived from the original equation by making the proper scal-
ings of the dependent or independent variables. In the following sections
we will treat our singular-perturbation problem, Eq.(36) and condition
(38), by the method of matched asymptotic expansions[22], [23].

(i) Outer Solution

Since ε^* is a small parameter, we expand the dependent variable
$u_1(\eta, \xi^*; \varepsilon^*)$ in an asymptotic series in powers of ε^*

$$u_1(\eta, \xi^*; \varepsilon^*) = v_o(\eta, \xi^*) + \varepsilon^* v_1(\eta, \xi^*) + \cdots \qquad (39)$$

Substitution of the expansion (39) into Eq.(36) generates a sequence
of approximate equations. The first equation of the sequence is

$$-v_o \frac{\partial v_o}{\partial \xi^*} + \frac{\partial v_o}{\partial \eta} + \frac{j}{2} \frac{v_o}{\eta} = 0 \qquad (40)$$

Eq.(40) is called the outer equation (or limit equation) and its solu-
tion is the first term of the asymptotic series (39). Introducing the
following transformation

$$v^* = v_o \, \eta^{j/2} \qquad , \qquad \eta^* = \int \eta^{j/2} \, d\eta \qquad (41)$$

into Eq.(40), we find that Eq.(40) becomes

$$\frac{\partial v^*}{\partial \eta} - v^* \frac{\partial v^*}{\partial \xi^*} = 0 \tag{42}$$

The general solution of Eq.(42) is

$$\xi^* + v^* \eta^* = g(v^*) \tag{43}$$

here $g(v^*)$ is a function to be determined from imposed conditions. We consider the N-wave as an example; however, the method used here is applicable to a general pulse-form and most of the results can easily be generalized to an arbitrary pulse. If the initial pulse is an N-wave, the initial condition (38) becomes

$$u_1(\eta, \xi^*) = \begin{cases} \frac{\varepsilon \Gamma}{\delta} \xi^* & |\xi^*| \leq \frac{\delta}{\varepsilon \Gamma} \\ 0 & |\xi^*| > \frac{\delta}{\varepsilon \Gamma} \end{cases} \quad \text{at} \quad \eta = 1 \tag{44}$$

Using the asymptotic series expansion for $u_1(\eta, \xi^*)$ and the transformation (41), we obtain the initial condition for Eq.(42)

$j = 1$:

$$v^*(2. \xi^*) = \begin{cases} \frac{\varepsilon \Gamma}{\delta} \xi^* & |\xi^*| \leq \frac{\delta}{\varepsilon \Gamma} \\ 0 & |\xi^*| > \frac{\delta}{\varepsilon \Gamma} \end{cases} \tag{45}$$

$j = 2$:

$$v^*(0, \xi^*) = \begin{cases} \frac{\varepsilon \Gamma}{\delta} \xi^* & |\xi^*| \leq \frac{\delta}{\varepsilon \Gamma} \\ 0 & |\xi^*| > \frac{\delta}{\varepsilon \Gamma} \end{cases} \tag{46}$$

(Note: $j = 1$ is the cylindrical case and $j = 2$ is the spherical case.)
Now the function $g(v^*)$ in Eq.(43) can be determined from the condition
(45) or (46); the results are

$j = 1$:

$$g(v^*) = 2v^* + \frac{\delta}{\varepsilon\rho} v^* \tag{47}$$

$j = 2$:

$$g(v^*) = \frac{\delta}{\varepsilon\rho} v^* \tag{48}$$

Thus, the general solution (43) becomes

$j = 1$:

$$v^*(\eta^*, \xi^*) = \frac{\xi^*}{\left(2 + \frac{\delta}{\varepsilon\rho} - \eta^*\right)} \tag{49}$$

$j = 2$:

$$v^*(\eta^*, \xi^*) = \frac{\xi^*}{\left(\frac{\delta}{\varepsilon\rho} - \eta^*\right)} \tag{50}$$

Expressing (49) and (50) in terms of original variables v_0 , η ,
ξ^* , we have the solution of outer equation (40)

$j = 1$:

$$v_0(\eta, \xi^*) = \frac{\xi^*}{\sqrt{\eta}\left(2 + \frac{\delta}{\varepsilon\rho} - 2\sqrt{\eta}\right)} \tag{51}$$

$j = 2$:

$$\mathcal{V}_0(\eta, \mathfrak{z}^*) = \frac{\mathfrak{z}^*}{\eta\left(\frac{\delta}{\varepsilon\Gamma} - \ln\eta\right)} \tag{52}$$

The solutions (51) and (52) both show that the velocity perturbation $\mathcal{V}_0$ is linearly proportional to $\mathfrak{z}^*$, hence, remains as N-shape, but the slope of the linear profile θ depends on η . As the pulse converges, i.e., η decreases, the slope of the linear profile increases. See Fig. 1.
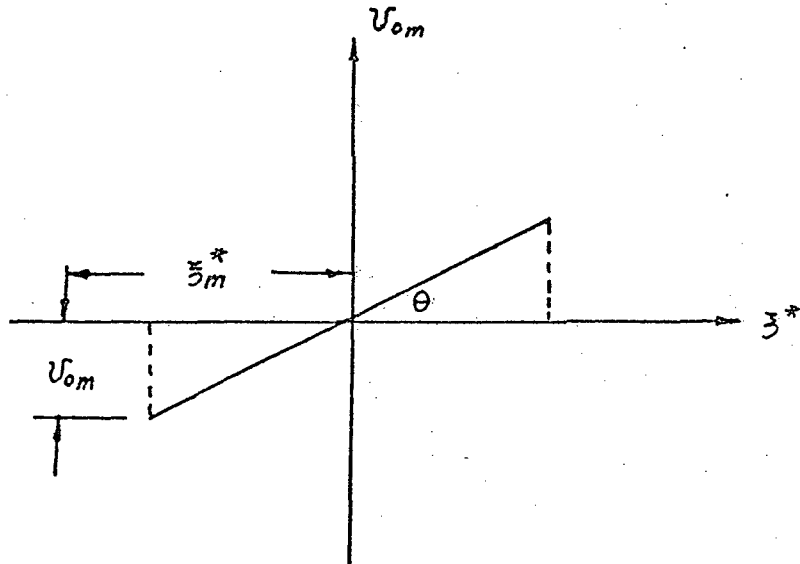

Fig. 1 Sketch of the solutions (51) and (52)

The maximum amplitude $\mathcal{V}_{0m}$ and pulse-length of N-wave $\mathfrak{z}_m^*$ (locations of the shocks) can not be determined by the solution (51) or (52) alone. The determination of these quantities requires an additional condition which is a weak version of the Rankine-Hugoniot relations for a weak shock wave. A simple method used by Seebass[25] to find such a condition is to

integrate Eq. (40) with respect to ξ^*

$$\int_{-\infty}^{\infty} \left(\frac{\partial \upsilon_0}{\partial \eta} + \frac{j}{2} \frac{\upsilon_0}{\eta} - \upsilon_0 \frac{\partial \upsilon_0}{\partial \xi^*} \right) d\xi^* = 0$$

Further arrangement yields

$$\frac{1}{\eta^{1/2}} \int_{-\infty}^{\infty} \frac{\partial}{\partial \eta} \left(\eta^{1/2} \upsilon_0 \right) d\xi^* - \left[\frac{1}{2} \upsilon_0^2 \right]_{-\infty}^{\infty} = 0$$

By the definition of a pulse, the disturbances vanish at $\xi^* = \pm\infty$, i.e., $\upsilon_0(\eta, \pm\infty) = 0$, and the above equation becomes

$$\frac{\partial}{\partial \eta} \int_{-\infty}^{\infty} \eta^{1/2} \upsilon_0 \, d\xi^* = 0$$

or equivalently

$$\int_{-\infty}^{\infty} \eta^{1/2} \upsilon_0 \, d\xi^* = C \tag{53}$$

where C is a constant.

For an initial N-wave at $\eta = 1$ as given by (44), Eq. (53) becomes

$$\eta^{1/2} \upsilon_{0m} \xi_m^* = \frac{\delta}{\varepsilon \Gamma} \tag{54}$$

We now have enough equations to determine the quantities υ_{0m} and ξ_m^*. Solving Eq. (51) and Eq. (54) with $j = 1$, we have

$j = 1$:

$$\xi_m^* = \pm \sqrt{\frac{\delta}{\epsilon \Gamma}} \sqrt{2 + \frac{\delta}{\epsilon \Gamma} - 2\sqrt{\eta}}$$

$$U_{o_m} = \frac{\sqrt{\frac{\delta}{\epsilon \Gamma}}}{\sqrt{\eta} \ \sqrt{2 + \frac{\delta}{\epsilon \Gamma} - 2\sqrt{\eta}}} \tag{55}$$

Similarly, from Eq.(52) and Eq.(54) with $j = 2$, we have

$j = 2$:

$$\xi_m^* = \pm \sqrt{\frac{\delta}{\epsilon \Gamma}} \ \sqrt{\frac{\delta}{\epsilon \Gamma} - \ln \eta}$$

$$U_{o_m} = \frac{\sqrt{\frac{\delta}{\epsilon \Gamma}}}{\eta \ \sqrt{\frac{\delta}{\epsilon \Gamma} - \ln \eta}} \tag{56}$$

These values ξ_m^* in (55) and (56) impose a restriction on the solution of outer equation (40), i.e., solutions (51) & (52) become

$j = 1$:

$$U_o(\eta, \xi^*) = \begin{cases} \dfrac{\xi^*}{\sqrt{\eta} \ (2 + \frac{\delta}{\epsilon \Gamma} - 2\sqrt{\eta})} & |\xi^*| \leq |\xi_m^*| \\[4mm] 0 & |\xi^*| > |\xi_m^*| \end{cases} \tag{57}$$

$j = 2$:

$$U_o(\eta, \xi^*) = \begin{cases} \dfrac{\xi^*}{\eta \ (\frac{\delta}{\epsilon \Gamma} - \ln \eta)} & |\xi^*| \leq |\xi_m^*| \\[4mm] 0 & |\xi^*| > |\xi_m^*| \end{cases} \tag{58}$$

On substituting of the expressions for $U_0(\eta, \zeta^*)$, (57) and (58), into the asymptotic series (39), we find the outer solution $u_1(\eta, \zeta^*; \varepsilon^*)$ to be

$j = 1$:

$$u_1(\eta, \zeta^*; \varepsilon^*) = \left\{ \begin{array}{c} \dfrac{\zeta^*}{\sqrt{\eta}\,(2 + \frac{\delta}{\varepsilon\eta} - 2\sqrt{\eta})} \\[12pt] 0 \end{array} \right\} + O(\varepsilon^*) \qquad \begin{array}{l} |\zeta^*| \leq |\zeta_m^*| \\[12pt] |\zeta^*| > |\zeta_m^*| \end{array} \qquad (59)$$

$j = 2$:

$$u_1(\eta, \zeta^*; \varepsilon^*) = \left\{ \begin{array}{c} \dfrac{\zeta^*}{\eta(\frac{\delta}{\varepsilon\eta} - \ln\eta)} \\[12pt] 0 \end{array} \right\} + O(\varepsilon^*) \qquad \begin{array}{l} |\zeta^*| \leq |\zeta_m^*| \\[12pt] |\zeta^*| > |\zeta_m^*| \end{array} \qquad (60)$$

(ii) Boundary-Layer Solution

In last section, we derived the outer equation from Eq.(36) and obtained the outer solution from it. In the process of deriving the outer equation (40), we assumed that $\varepsilon^* \frac{\partial^2 U_0}{\partial \zeta^*}$ is of $O(\varepsilon^*)$, consequently $\frac{\partial^2 U_0}{\partial \zeta^{*2}} = O(1)$ is assumed. The outer solutions (57) and (58) describe an N-wave with discontinuous wavefronts at $\pm \zeta_m^*$. In the neighborhood of these wavefronts $\frac{\partial}{\partial \zeta^*}$ and $\frac{\partial^2}{\partial \zeta^{*2}}$ will be large, so that the term $\varepsilon^* \frac{\partial^2 U_0}{\partial \zeta^{*2}}$ is no longer $O(\varepsilon^*)$. The regions where $\frac{\partial}{\partial \zeta}$ and $\frac{\partial^2}{\partial \zeta^{*2}}$ are large, the boundary-layers, need separate consideration. We expect the boundary-layer regions to be narrow regions with large gradients, so that we may introduce a boundary-layer coordinate by which a boundary-layer equation is derived for each region. In what follows

we will treat the boundary-layer region in detail for the cylindrical case.
The same analysis can be easily extended to the spherical case.

(a) $j = 1$ (cylindrical case) :

From the solution (55) we know that the boundary layers are located

at $\xi^* = \pm \sqrt{\frac{\delta}{\varepsilon\Gamma}} \sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}$.   To investigate the boundary layer at

$\xi^* = - \sqrt{\frac{\delta}{\varepsilon\Gamma}} \sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}$ we introduce the following boundary-layer

coordinates:

$$\bar{\xi} = \frac{\xi^* + \sqrt{\frac{\delta}{\varepsilon\Gamma}} \sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}}{\psi(\varepsilon^*)}$$

(61)

$$\eta = \eta$$

Here $\psi(\varepsilon^*)$ characterizes the width of the boundary layer.   Transform-
ing the approximate equation of converging region, Eq.(36), into the
boundary-layer coordinates, we have

$$\frac{\partial u_1}{\partial \eta} - \frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\psi(\varepsilon^*)\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}} \frac{\partial u_1}{\partial \bar{\xi}} + \frac{j}{2}\frac{u_1}{\eta} - \frac{u_1}{\psi(\varepsilon^*)}\frac{\partial u_1}{\partial \bar{\xi}} = -\frac{\varepsilon^*}{\psi^2(\varepsilon^*)}\frac{\partial^2 u_1}{\partial \bar{\xi}^2}$$

(62)

Next we expand the dependent variable $u_1(\eta, \bar{\xi} ; \varepsilon^*)$ in an asymptotic
series in powers of ε^*

$$u_1(\eta, \bar{\xi} ; \varepsilon^*) = v_B(\eta, \bar{\xi}) + \varepsilon^* v_{B1}(\eta, \bar{\xi}) + \cdots$$

(63)

Substitution of the expansion (63) into Eq.(62) yields a sequence of
approximate equations among which the first one is

$$v_B \frac{\partial v_B}{\partial \bar{\xi}} + \frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}} \frac{\partial v_B}{\partial \bar{\xi}} = \frac{\partial^2 v_B}{\partial \bar{\xi}^2}$$

(64)

provided $\psi(\varepsilon^*) = \varepsilon^*$.

Equation (64) is called the boundary-layer equation and governs the flow field in the boundary-layer region about $\xi^* = -\sqrt{\frac{\delta}{\varepsilon \Gamma}}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}$. The solution of this equation must match the outer solution (57) at one side and match the undisturbed state at other side; thus the solution must satisfy the following conditions:

$$v_B, \quad \frac{\partial v_B}{\partial \bar{\xi}} \sim 0 \qquad \text{as} \qquad \bar{\xi} \to -\infty$$

$$v_B \sim \frac{-\sqrt{\frac{\delta}{\varepsilon \Gamma}}}{\sqrt{\eta}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}} \qquad \text{as} \qquad \bar{\xi} \to +\infty \tag{65}$$

Integrating Eq.(64) twice and using the conditions (65) to determine the constants of integration, we obtain the solution of boundary-layer equation (64)

$$v_B(\eta,\bar{\xi}) = \frac{-\sqrt{\frac{\delta}{\varepsilon \Gamma}}}{2\sqrt{\eta}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}}\left[1+\tanh\left(\frac{\bar{\xi}\sqrt{\frac{\delta}{\varepsilon \Gamma}}}{4\sqrt{\eta}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}}\right)\right] \tag{66}$$

Upon substituting the above expression for $v_B(\eta,\bar{\xi})$ in the asymptotic series (63), we have the boundary-layer-solution at $\xi^* = -\sqrt{\frac{\delta}{\varepsilon \Gamma}}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}$

$$u_1(\eta,\bar{\xi};\varepsilon^*) = \frac{-\sqrt{\frac{\delta}{\varepsilon \Gamma}}}{2\sqrt{\eta}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}}\left[1+\tanh\left(\frac{\bar{\xi}\sqrt{\frac{\delta}{\varepsilon \Gamma}}}{4\sqrt{\eta}\sqrt{2+\frac{\delta}{\varepsilon \Gamma}-2\sqrt{\eta}}}\right)\right] + O(\varepsilon^*) \tag{67}$$

The boundary-layer solution at other discontinuity of the N-wave,

$\xi^* = + \sqrt{\frac{\delta}{\varepsilon P}} \sqrt{2 + \frac{\delta}{\varepsilon P} - 2\sqrt{\eta}}$, can be found in a similar manner. The boundary-layer coordinates are defined as

$$\hat{\xi} = \frac{\xi^* - \sqrt{\frac{\delta}{\varepsilon P}} \sqrt{2 + \frac{\delta}{\varepsilon P} - 2\sqrt{\eta}}}{\varepsilon^*}$$

$$\eta = \eta \tag{68}$$

The dependent variable $u_1(\eta , \hat{\xi} ; \varepsilon^*)$ is expanded in an asymptotic series in powers of ε^*

$$u_1(\eta, \hat{\xi} ; \varepsilon^*) = U_B(\eta, \hat{\xi}) + \varepsilon^* U_{B_1}(\eta, \hat{\xi}) + \cdots \tag{69}$$

Transforming the approximate equation of converging region, Eq.(36), in terms of the boundary-layer variables (68) and substituting the asymptotic series (64) into the resultant equation, we have a sequence of approximate equations; the first equation among them is

$$U_B \frac{\partial U_B}{\partial \hat{\xi}} - \frac{\sqrt{\frac{\delta}{\varepsilon P}}}{2 \sqrt{\eta} \sqrt{2 + \frac{\delta}{\varepsilon P} - 2\sqrt{\eta}}} \frac{\partial U_B}{\partial \hat{\xi}} = \frac{\partial^2 U_B}{\partial \hat{\xi}^2} \tag{70}$$

The boundary conditions for this equation are

$$U_B , \frac{\partial U_B}{\partial \hat{\xi}} \sim O \qquad \text{as} \qquad \hat{\xi} \to \infty$$

$$\tag{71}$$

$$U_B \sim \frac{\sqrt{\frac{\delta}{\varepsilon P}}}{\sqrt{\eta} \sqrt{2 + \frac{\delta}{\varepsilon P} - 2\sqrt{\eta}}} \qquad \text{as} \qquad \hat{\xi} \to -\infty$$

The solution of Eq.(70) satisfying the boundary conditions (71) is

$$v_B(\eta, \hat{\xi}) = \frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}} \left[1 + \tanh\left(\frac{\hat{\xi}\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{4\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}}\right)\right] \qquad (72)$$

and consequently the boundary-layer solution at $\xi^* = \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}$ by (69) is

$$u_1(\eta, \hat{\xi}; \varepsilon^*) = \frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}} \left[1 - \tanh\left(\frac{\hat{\xi}\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{4\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}}\right)\right] + O(\varepsilon^*)$$

$$(73)$$

(b) $j = 2$ (spherical case) :

The previous treatment of the cylindrical case can be easily extended to the spherical case. The boundary layers of the spherical N-wave are at $\xi^* = \pm\sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}$. Following the same procedure as before we find that the boundary-layer solution at $\xi^* = -\sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}$ is

$$u_1(\eta, \bar{\xi}; \varepsilon^*) = -\frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\eta\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}} \left[1 + \tanh\left(\frac{\bar{\xi}\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{4\eta\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}}\right)\right] + O(\varepsilon^*)$$

$$(74)$$

where $$\bar{\xi} = \frac{\xi^* + \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}}{\varepsilon^*}$$

$$(75)$$

and the boundary-layer solution at $\xi^* = \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}$ is

$$u_1(\eta, \hat{\zeta}; \varepsilon^*) = \frac{\sqrt{\frac{\delta}{\varepsilon \Gamma}}}{2\eta \sqrt{\frac{\delta}{\varepsilon \Gamma} - \ln \eta}} \left[1 - \tanh\left(\frac{\hat{\zeta} \sqrt{\frac{\delta}{\varepsilon \Gamma}}}{4\eta \sqrt{\frac{\delta}{\varepsilon \Gamma} - \ln \eta}} \right) \right] + O(\varepsilon^*) \tag{76}$$

where
$$\hat{\zeta} = \frac{\zeta^* - \sqrt{\frac{\delta}{\varepsilon \Gamma}} \sqrt{\frac{\delta}{\varepsilon \Gamma} - \ln \eta}}{\varepsilon^*} \tag{77}$$

(iii) Uniformly Valid Composite Solution

We have obtained the outer solution and boundary-layer solutions of Eq.(36); each of them is valid in its own domain and nowhere else. However, a uniformly valid composite solution (first term of a composite expansion) can be constructed from the outer and boundary-layer solutions. The form of the composite solution is not unique, since it depends on the manner of its construction. In this problem we prefer to use the method of multiplicative composition[23] because the resultant composite solution has simpler form. This method states that the outer expansion is to be multiplied by a correction factor consisting of the ratio of the boundary-layer expansion to its outer expansion, or the boundary-layer expansion is treated similarly. This gives

$$f^{(m,n)} = f_o^{(n)} \frac{f_b^{(m)}}{\left[f_b^{(m)} \right]_o^{(n)}} = f_b^{(m)} \frac{f_o^{(n)}}{\left[f_o^n \right]_b^{(m)}} = \frac{f_o^{(n)} f_b^{(m)}}{\left[f_o^n \right]_i^{(m)} = \left[f_b^m \right]_o^{(n)}} \tag{78}$$

Here f_c is the composite expansion; f_o is the outer expansion; f_b is the boundary-layer expansion; m , n are the number of terms in the expansion series. In our problem we have the outer solution and

boundary-layer solution, i.e., only first term of the expansion, therefore $m = n = 1$. Thus, the formula (78) simply becomes

$$u_{1c} = \frac{(\text{ outer solution })(\text{ boundary-layer solution })}{(\text{ outer solution in boundary-layer variables and } \varepsilon^* \to 0)}$$

(79)

We construct the composite solution for each boundary-layer location according to the formula (79) and the combination of two composite solutions gives the composite solution for whole N-wave as follows:

$j = 1$:

$$U_{1c}(\eta, \xi^*; \varepsilon^*) = \frac{\xi^*}{2\sqrt{\eta}\,(2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta})} \left[\tanh\left(\frac{\xi^* + \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}}{4\varepsilon^*\sqrt{\frac{\varepsilon\Gamma}{\delta}}\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}} \right) \right.$$

$$\left. - \tanh\left(\frac{\xi^* - \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}}{4\varepsilon^*\sqrt{\frac{\varepsilon\Gamma}{\delta}}\sqrt{\eta}\sqrt{2 + \frac{\delta}{\varepsilon\Gamma} - 2\sqrt{\eta}}} \right) \right] + O(\varepsilon^*)$$

(80)

$j = 2$:

$$U_{1c}(\eta, \xi^*; \varepsilon^*) = \frac{\xi^*}{2\eta\,(\frac{\delta}{\varepsilon\Gamma} - \ln\eta)} \left[\tanh\left(\frac{\xi^* + \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}}{4\varepsilon^*\eta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}} \right) \right.$$

$$\left. - \tanh\left(\frac{\xi^* - \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\eta}}{4\varepsilon^*\eta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}} \right) \right] + O(\varepsilon^*)$$

(81)

It can be verified that each composite solution (80) or (81) reduces to the outer solution, or boundary-layer solutions, depending on the

appropriate scaling of the independent variables.

Thus, for the weak nonlinear case in which the governing equation (36) is a radial Burgers equation with a small parameter ε^* , we have shown that a composite solution, i.e., the first term of composite expansion, can be obtained. These composite solutions describe a converging N-wave in a first-order approximation. The structure of the front and rear shocks of the N-wave is clearly illustrated by the composite solutions. These solutions (80) and (81) are sketched in Fig. 2 and Fig. 3 respectively.
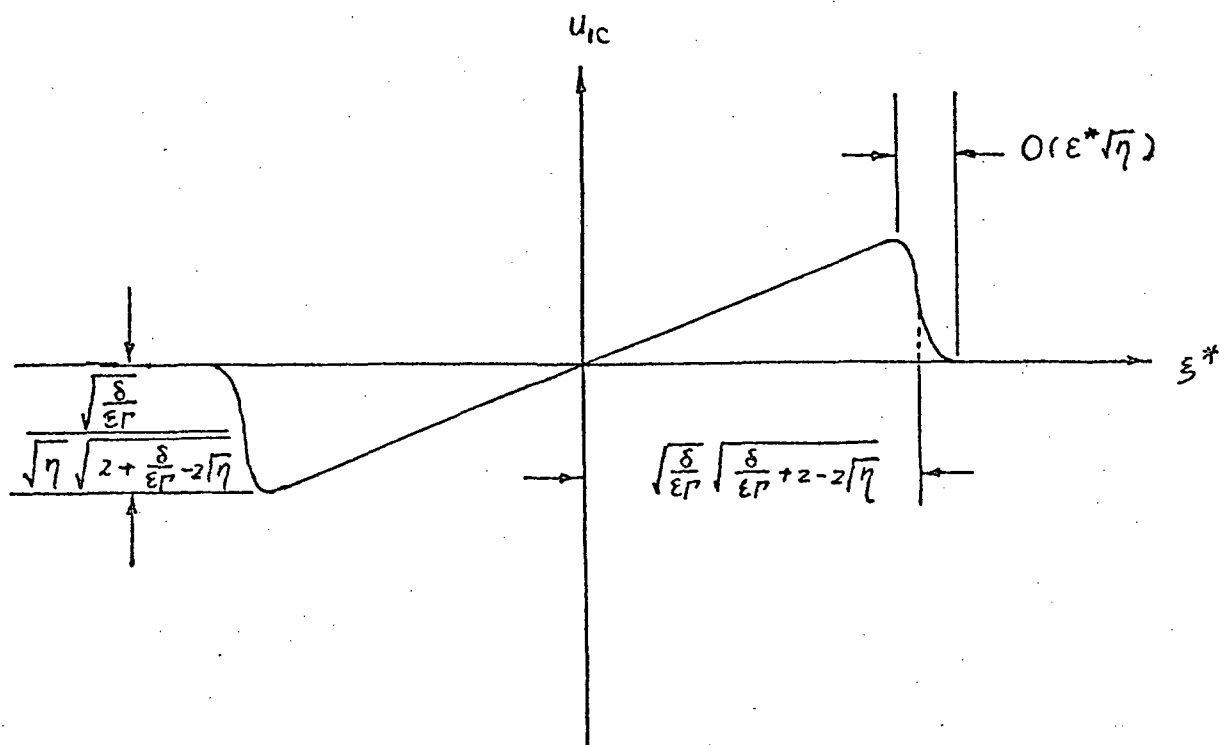

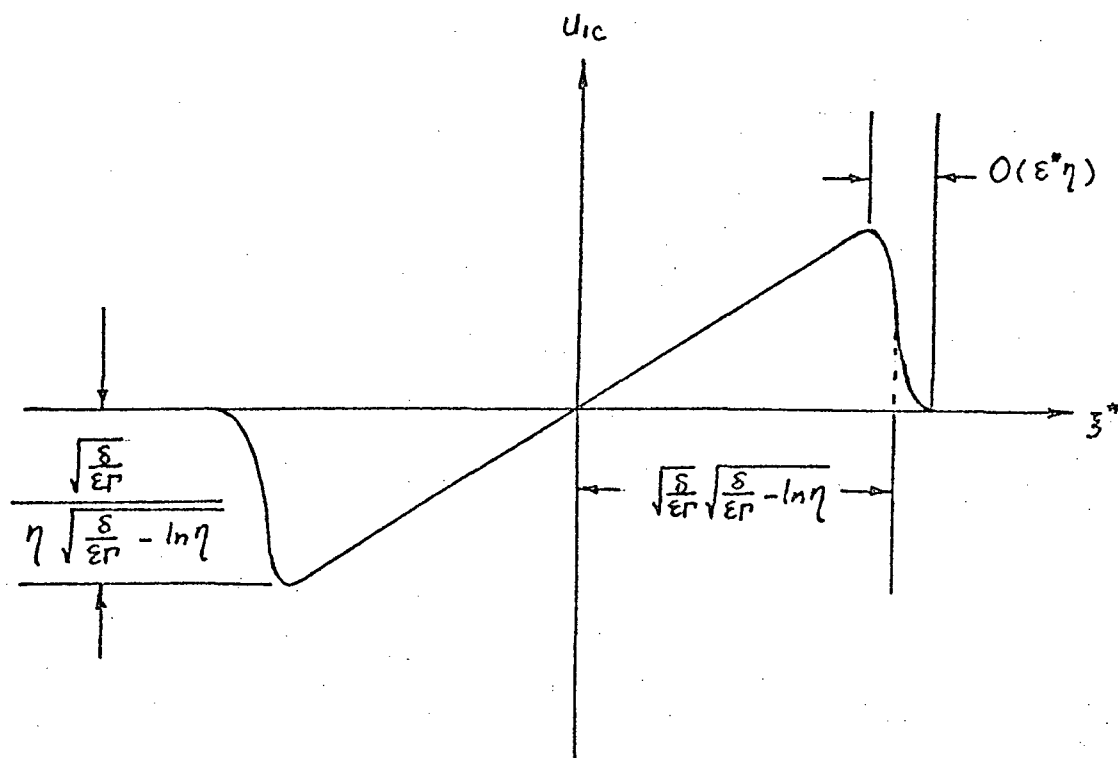Fig. 2   Profile of the converging N-wave in the cylindrical case

Fig. 3 Profile of the converging N-wave in the spherical case

IV. INTERACTION REGION

The motion of the converging pulse in the converging region $r \gg \delta$ has been studied in Chapter III. In this chapter we investigate the converging pulse as it enters in the interaction region $r \leq \delta$. In the derivation of the approximate equation for the converging region, Eq.(21), we have used the condition $r \gg \delta$; furthermore, Eq.(21) contains only a first-order wave operator which is not capable of handling waves moving in both directions. Consequently Eq.(21) does not apply to the interaction region defined by $r \leq \delta$, which contains the axis or center of symmetry. Since the size of the interaction region is of the same order as the pulse-length of the converging pulse, it is conceivable that the motion of the converging pulse in the interaction region is a short-time process in comparison with that in the converging region. We expect that the relative importance of each factor (i.e., geometrical, nonlinear, and viscous and heat-conducting effects) in the interaction region will be different from that in the converging region.

The basic equations are still the nondimensional equations (8) to (12). We scale the independent variables, space and time, according to the dimension of the interaction region δ as follows:

$$x = \frac{r}{\delta} \qquad\qquad \tau = \frac{t}{\delta}$$

Here r , t are the primed quantities defined by (7).

Changing the independent variables (r, t) in Eqs.(8) to (12) into the variables (x, τ) as defined by (82), we have the basic equations in terms of the variables (x, τ)

$$\frac{\partial \rho}{\partial \tau} + u \frac{\partial \rho}{\partial x} + (1+\rho) \frac{\partial u}{\partial x} = -j(1+\rho)\frac{u}{x} \tag{83}$$

$$(1+\rho)\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} = -\frac{1}{\gamma}\frac{\partial p}{\partial x} + \frac{1}{Re_\lambda}\left[\frac{\partial^2 u}{\partial x^2} + \frac{j}{\partial x}\frac{\partial u}{\partial x} - j\frac{u}{x^2}\right] \tag{84}$$

$$(1+\rho)(1+T)\left(\frac{\partial S}{\partial \tau} + u \frac{\partial S}{\partial x}\right) = \frac{K}{Re_\lambda}\left[\frac{\partial^2 T}{\partial x^2} + \frac{j}{x}\frac{\partial T}{\partial x}\right]$$

$$+ \frac{\gamma(\gamma-1)}{Re_\lambda}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{u}{x}\right)^2 - j\frac{u}{x}\frac{\partial u}{\partial x}\right] \tag{85}$$

$$(1+P) = e^{S}(1+\rho)^{\gamma} \tag{86}$$

$$(1+P) = (1+\rho)(1+T) \tag{87}$$

where Re_λ is the Reynolds number based on pulse-length λ ; the third fundamental assumption (18) states that $Re_\lambda \gg 1$.
The boundary conditions imposed on the system of equations (83) to (87) are that the solution should have no motion at the axis or center of symmetry and should match the solution of the converging region as
$x \longrightarrow \infty$, $\frac{\tau}{x} \longrightarrow -1$.

We will study the interaction region in the linear and weak non-linear cases separately because they are based on different assumptions concerning the order of the parameters ε , δ and Re_λ , which certainly will affect the final form of the approximate equation for the

interaction region. The linear case is to be studied first.

Linear Case

In Chapter III, we have obtained the solutions of the converging

region in the linear case (Eqs.32 and 34). These solutions show that as

the converging pulse approaches the interaction region, the amplitude of

the pulse becomes $O\left(\dfrac{\varepsilon}{\delta^{3/2}}\right)$ and pulse-form becomes smooth in the sense

that the gradients are less or equal to $O(1)$ as $\eta \to \delta$. Hence, we

may assume that

$$\frac{\partial}{\partial \tau} \sim \frac{\partial}{\partial x} \sim \frac{\partial^2}{\partial x^2} \sim O(1) \tag{88}$$

are valid everywhere on the converging pulse in the interaction region.

Since $\dfrac{\varepsilon}{\delta^{3/2}} \ll 1$ in the linear case, we may expand the dependent vari-

ables in power series of ε

$$u(x,\tau;\varepsilon) = \varepsilon\, u_1(x,\tau) + \varepsilon^2 u_2(x,\tau) + \cdots$$

$$p(x,\tau;\varepsilon) = \varepsilon\, P_1(x,\tau) + \varepsilon^2 P_2(x,\tau) + \cdots$$

$$\rho(x,\tau;\varepsilon) = \varepsilon\, \rho_1(x,\tau) + \varepsilon^2 \rho_2(x,\tau) + \cdots$$

$$T(x,\tau;\varepsilon) = \varepsilon\, T_1(x,\tau) + \varepsilon^2 T_2(x,\tau) + \cdots$$

$$S(x,\tau;\varepsilon) = \varepsilon\, S_1(x,\tau) + \varepsilon^2 S_2(x,\tau) + \cdots$$

$$\tag{89}$$

Upon substituting these series expansions (89) into the basic equations (83) to (87) and making use of the relation (88), one obtains a sequence of approximate equations of which the first-order equations are

$$\frac{\partial \rho_1}{\partial \tau} + \frac{\partial u_1}{\partial x} + j\,\frac{u_1}{x} = 0$$

$$\frac{\partial u_1}{\partial \tau} + \frac{1}{\gamma}\,\frac{\partial p_1}{\partial x} = 0$$

$$\frac{\partial S_1}{\partial \tau} = 0$$

$$p_1 - \rho_1 = T_1$$

$$p_1 - \gamma \rho_1 = S_1 \tag{90}$$

which, after some manipulation, yield a single equation for u_1

$$\frac{\partial^2 u_1}{\partial \tau^2} = \frac{\partial^2 u_1}{\partial x^2} + j\left(\frac{1}{x}\,\frac{\partial u_1}{\partial x} - \frac{u_1}{x^2}\right) \tag{91}$$

and

$$S_1 = 0$$

$$p_1 = \gamma \rho_1$$

$$\frac{\partial u_1}{\partial \tau} + \frac{\partial \rho_1}{\partial x} = 0 \tag{92}$$

The flow field in the interaction region in the linear case satisfies to first order the linear wave equation with radial symmetry (91). The inner and outer boundary conditions for the solution of Eq.(91) are that the solution should vanish at the axis or center of symmetry for all time and should match the solution of the converging region as $x \to \infty$, $\frac{\tau}{x} \to -1$. The explicit form of the inner boundary condition is simply

$$u_{1}(x,\tau) = 0 \qquad \text{at} \quad x = 0 \quad \text{for all } \tau$$

and the explicit form of the outer boundary condition is derived from the solution of the converging region Eq.(32) and Eq.(34), depending on the initial pulse-form used. Expressing the solutions (32) and (34) in terms of the variables of the interaction region x, τ

$$\eta = r = x\delta$$

$$\xi = \frac{t+r}{\delta} = \tau + x \qquad\qquad (94)$$

and expanding the resultant expressions for small δ while τ , x and δRe are fixed, we have the outer boundary condition for Eq.(91)

$$u_{1}(x,\tau) \underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{\sim} \frac{1}{\delta^{3/2} x^{3/2}} \frac{\tau + x}{\left(1 + \frac{4}{\delta Re}\right)^{3/2}} e^{-\dfrac{(\tau+x)^{2}}{1+\frac{4}{\delta Re}}} \qquad (95)$$

or

$$u_{1}(x,\tau) \underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{\sim} \frac{1}{\delta^{3/2} x^{3/2} \sqrt{\delta Re}} \left\{ \frac{1}{\sqrt{\pi}} \left(e^{-\delta Re\left(\frac{\tau+x+1}{2}\right)^{2}} - e^{-\delta Re\left(\frac{\tau+x-1}{2}\right)^{2}} \right) \right.$$

$$\left. + \sqrt{\delta Re}(\tau+x) \left[\Phi\left(\frac{\sqrt{\delta Re}\,(\tau+x+1)}{\sqrt{2}}\right) - \Phi\left(\frac{\sqrt{\delta Re}\,(\tau+x-1)}{\sqrt{2}}\right) \right] \right\} \qquad (96)$$

The mathematical problem for the interaction region is to find the solution

of Eq.(91) with boundary conditions (93) and (95) (or 96). It is neces-
sary to solve the cylindrical and spherical cases separately.

(a) j = 1 (cylindrical case) :

The method of Fourier transforms is used to solve Eq.(91) and bound-
ary conditions (93) and (95) (or 96) in which j = 1 . Taking Fourier
transform with respect to τ of Eq.(91) and boundary conditions (93)
and (95), we have

$$\frac{\partial^2 \hat{u}_1}{\partial x^2} + \frac{1}{x} \frac{\partial \hat{u}_1}{\partial x} + (s^2 - \frac{1}{x}) \, \hat{u}_1 \; = \; 0 \qquad\qquad (97)$$

where

$$\hat{u}_1 (x, s) = \int_{-\infty}^{\infty} u_1 (x, \tau) \, e^{-is\tau} \, d\tau \qquad\qquad (98)$$

and the boundary conditions become

$$\hat{u}_1 (x, s) \; = \; \frac{1}{\sqrt{\delta}} \, \frac{1}{\sqrt{x}} \, e^{isx} \frac{\sqrt{\pi}}{2i} \, s \, e^{-\frac{1}{4} (1 + \frac{4}{\delta Re}) s^2} \qquad\qquad (99)$$

$$\hat{u}_1 (x, s) \; = \; 0 \qquad\qquad \text{at} \qquad x = 0 \qquad\qquad (100)$$

The general solution of Eq.(97) is

$$\hat{u}_1 (x, s) = C_1 (s) \, H_1^{(1)} (xs) \; + \; C_2 (s) \, H_1^{(2)} (xs) \qquad\qquad (101)$$

where $H_1^{(1)} (xs)$ and $H_1^{(2)} (xs)$ are the Hankel functions of the first
and second kinds with the following special properties:

$$H_1^{(1)}(z) = J_1(z) + i \, Y_1(z) \longrightarrow \frac{z}{2} + i\left(-\frac{2}{\pi}\frac{1}{z}\right) \qquad \text{as} \qquad |z| \to 0$$

$$\text{(102)}$$

$$H_1^{(2)}(z) = J_1(z) - i \, Y_1(z) \longrightarrow \frac{z}{2} - i\left(-\frac{2}{\pi}\frac{1}{z}\right) \qquad \text{as} \qquad |z| \to 0$$

and

$$H_1^{(1)}(z) \sim \sqrt{\frac{z}{\pi z}} \, e^{i\left(z-\frac{3}{4}\pi\right)} \left[1 + O\left(\frac{1}{|z|}\right)\right] \qquad \begin{array}{c} -\pi < \arg z < 2\pi \\ \text{as} \quad |z| \to \infty \end{array}$$

$$\text{(103)}$$

$$H_1^{(2)}(z) \sim \sqrt{\frac{z}{\pi z}} \, e^{-i\left(z-\frac{3}{4}\pi\right)} \left[1 + O\left(\frac{1}{|z|}\right)\right] \qquad \begin{array}{c} -2\pi < \arg z < \pi \\ \text{as} \quad |z| \to \infty \end{array}$$

The functions $C_1(s)$ and $C_2(s)$ in the general solution (101) are to be determined from the boundary conditions (99) and (100) in which the properties of Hankel functions (102) and (103) are used. From the boundary condition (100) and the relations (102) we have

$$C_2(s) = C_1(s)$$

$$\text{(104)}$$

From the outer boundary condition (99) and the relations (103) we find

$$C_1(s) = \frac{1}{\sqrt{\pi}} \frac{\pi}{2\sqrt{2}} \, e^{\frac{1}{4}\pi i} \sqrt{s} \, s \, e^{-\frac{1}{4}\left(1+\frac{4}{sRe}\right)s^2} \qquad -\pi < \arg s < 2\pi$$

$$\text{(105)}$$

Application of the inverse Fourier transform to the general solution (101) in which $C_1(s)$ and $C_2(s)$ are given by (104) and (105) yields the required solution

$$u_1(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(x,s) \, e^{is\tau} \, ds$$

$$\text{(106)}$$

An attempt to evaluate the complete inverse Fourier integral is without success; however the asymptotic behavior for large x and limiting behavior for small x are obtained and discussed.

Asymptotic Behavior for Large x :

The Hankel functions in $\hat{u}_1(x,s)$ of (106) are replaced by their asymptotic forms (103); then, evalulating the resultant integral (106), we have the asymptotic solution for large x

$$u_1(x,\tau) \underset{x \to \infty}{\sim} \frac{1}{\sqrt{\delta}} \frac{1}{\sqrt{x}} \frac{\tau+x}{(1+\frac{4}{\delta Re})^{3/2}} e^{-\frac{(\tau+x)^2}{1+\frac{4}{\delta Re}}} - \frac{1}{\sqrt{\delta}} \frac{1}{\sqrt{x}} \frac{\tau-x}{(1+\frac{4}{\delta Re})^{3/2}} e^{-\frac{(\tau-x)^2}{1+\frac{4}{\delta Re}}}$$

$$+ O\left(\frac{1}{x^{3/2}\sqrt{\delta}(1+\frac{4}{\delta Re})^{3/2}}\right)$$

$$(107)$$

It is easily seen that the second term in (107) is exponentially small as $x \to \infty$, $\frac{\tau}{x} \to -1$ and the asymptotic solution (107) matches the outer boundary condition (95). On the other hand as $x \to \infty$, $\frac{\tau}{x} \to +1$, the first term in (107) is exponentially small, and the asymptotic solution (107) predicts the reflected outgoing pulse.

Limiting Solution for Small x :

Using the relations (102) for the Hankel functions in $\hat{u}_1(x,s)$ in (106), we have the solution for small x ,

$$u_1(x,\tau) \underset{x \to 0}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} C_1(s) 2 J_1(xs) \underset{x \to 0}{e^{is\tau}} ds$$

$$= -\frac{x}{\sqrt{\delta}}\frac{\sqrt{\pi}}{\sqrt{2}}\frac{1}{\left(1+\frac{4}{\delta Re}\right)^{7/4}} \exp\left(-\frac{\tau^2}{2\left(1+\frac{4}{\delta Re}\right)}\right) D_{5/2}\left(\frac{-\tau}{\sqrt{2}\sqrt{1+\frac{4}{\delta Re}}}\right) + O\left(\frac{x^3}{\sqrt{\delta}\left(1+\frac{4}{\delta Re}\right)^{7/2}}\right)$$

(108)

where $D_{5/2}(z)$ is the parabolic cylinder function of order $5/2$.
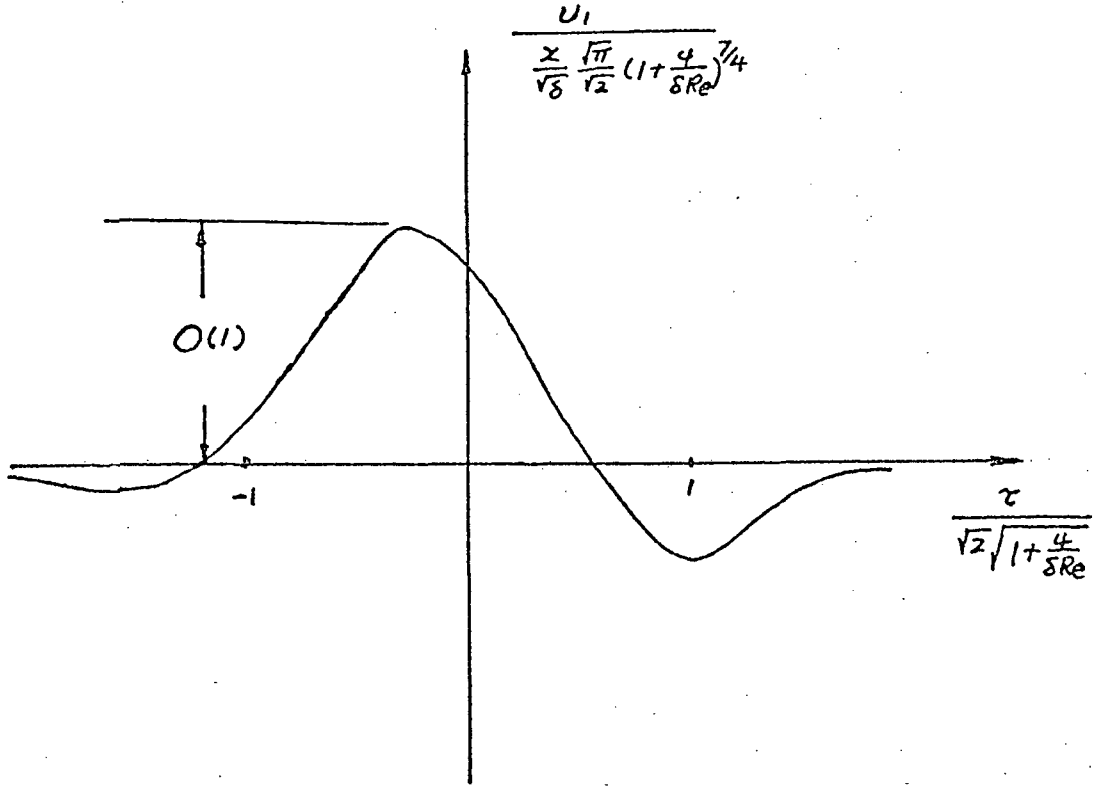

Fig. 4 Velocity perturbation near and at the axis (Eq.(108))

Knowing the velocity perturbation, we can easily calculate the pressure and density perturbations from Eq.(92); the results are

$$\rho_1(x,\tau) = \frac{2^{3/4}\sqrt{\pi}}{\sqrt{\delta}\left(1+\frac{4}{\delta Re}\right)^{7/4}}\int_{-\infty}^{\infty} \exp\left(-\frac{\tau^2}{2\left(1+\frac{4}{\delta Re}\right)}\right) D_{5/2}\left(\frac{-\tau}{\sqrt{2}\sqrt{1+\frac{4}{\delta Re}}}\right) d\tau + O\left(\frac{x^2}{\sqrt{\delta}\left(1+\frac{4}{\delta Re}\right)^{3/4}}\right)$$

$x \to 0$

(109)

and
$$P_1(x,\tau) = \gamma \rho_1(x,\tau)$$
$$\quad x \to 0 \qquad\qquad x \to 0$$
(110)

The limiting solutions show that in first-order approximation the velocity perturbation is linearly proportional to x as $x \to 0$ and the pressure and density perturbations are independent of x as $x \to 0$, varying with τ only. It should be noted that the pressure and density perturbations are $O(\frac{1}{\sqrt{\delta}})$ even at the axis of symmetry; the perturbations remain small in the interaction region, hence small in the whole flow field in the linear case.

If we apply the same analysis to the Eq.(91) with other outer boundary condition (96), we can also obtain the asymptotic solution for large x

$$u_1(x,\tau) \underset{x \to \infty}{\sim} \frac{1}{\sqrt{\delta}} \frac{1}{\sqrt{x}} \left[G(\tau+x) - G(\tau-x) \right] + O\left(\frac{1}{\sqrt{\delta}\, x^{3/2}}\right)$$

and limiting solutions for small x
(111)

$$u_1(x,\tau) = \frac{x}{6\sqrt{2}\,\sqrt{\delta}} \frac{d^2}{d\tau^2} \int_{-\infty}^{\tau} \frac{G(\alpha)}{\sqrt{\tau-\alpha}}\, d\alpha + O\left(\frac{x^3}{\sqrt{\delta}}\right)$$
$$\quad x \to 0$$
(112)

$$\rho_1(x,\tau) = \frac{1}{6\sqrt{2}\,\sqrt{\delta}} \frac{d}{d\tau} \int_{-\infty}^{\tau} \frac{G(\alpha)}{\sqrt{\tau-\alpha}}\, d\alpha + O\left(\frac{x^2}{\sqrt{\delta}}\right)$$
$$\quad x \to 0$$
(113)

$$P_1(x,\tau) = \gamma \rho_1(x,\tau)$$
$$\quad x \to 0 \qquad\qquad x \to 0$$
(114)

Here the function $G(z)$ has the form

$$G(z) = \left\{ z \left[\Phi\left(\frac{\sqrt{\delta Re}\,(z+1)}{\sqrt{2}} \right) - \Phi\left(\frac{\sqrt{\delta Re}\,(z-1)}{\sqrt{2}} \right) \right] \right.$$
$$\left. + \frac{1}{\sqrt{\delta Re}} \frac{1}{\sqrt{\pi}} \left(e^{-\delta Re\left(\frac{z+1}{2}\right)^2} - e^{-\delta Re\left(\frac{z-1}{2}\right)^2} \right) \right\} \tag{115}$$

(b) $j = 2$ (spherical case) :

The mathematical problem we intend to solve is Eq.(91) with the boundary conditions (93) and (95) (or 96), in which $j = 2$. Applying the Fourier transform to Eq.(91) and boundary conditions (93) and (95), we find

$$\frac{\partial^2 \hat{u}_1}{\partial x^2} + \frac{2}{x} \frac{\partial \hat{u}_1}{\partial x} + \left(s^2 - \frac{2}{x} \right) \hat{u}_1 = 0 \tag{116}$$

where

$$\hat{u}_1(x,s) = \int_{-\infty}^{\infty} u_1(x,\tau) e^{-is\tau} d\tau \tag{117}$$

and the boundary conditions are

$$\hat{u}_1(x,s) = 0 \qquad \text{at} \quad x = 0 \tag{118}$$

$$\hat{u}_1(x,s) \underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{=} \frac{e^{isx}}{\delta x} \frac{\sqrt{\pi}}{2i} s e^{-\frac{1}{4}\left(1 + \frac{4}{\delta Re}\right)s^2} \tag{119}$$

The general solution of Eq.(116) is

$$\hat{u}_1(x,s) = C_1(s) \sqrt{\frac{\pi}{2xs}} H_{3/2}^{(1)}(xs) + C_2(s) \sqrt{\frac{\pi}{2xs}} H_{3/2}^{(2)}(xs) \tag{120}$$

where $H_{3/2}^{(1)}(z)$ and $H_{3/2}^{(2)}(z)$ are the Hankel functions of first and second kinds of order $3/2$; they have the following properties:

$$H_{3/2}^{(1)}(z) = \sqrt{\frac{2z}{\pi}} \left(\frac{z}{3} + i\frac{1}{z^2} \right) \qquad \text{as} \qquad |z| \to 0$$

$$H_{3/2}^{(2)}(z) = \sqrt{\frac{2z}{\pi}} \left(\frac{z}{3} - i\frac{1}{z^2} \right) \qquad \text{as} \qquad |z| \to 0 \tag{121}$$

and

$$H_{3/2}^{(1)}(z) \sim \sqrt{\frac{z}{\pi z}}\, e^{i(z-\pi)} \left[1 + O\left(\frac{1}{|z|}\right) \right] \qquad \text{as} \qquad |z| \to \infty$$

$$H_{3/2}^{(2)}(z) \sim \sqrt{\frac{z}{\pi z}}\, e^{-i(z-\pi)} \left[1 + O\left(\frac{1}{|z|}\right) \right] \qquad \text{as} \qquad |z| \to \infty \tag{122}$$

The functions $C_1(s)$ and $C_2(s)$ in the general solution (120) are determined by using the boundary conditions (118) and (119) and the properties of Hankel functions (121), (122); the results are

$$C_2(s) = C_1(s) \tag{123}$$

$$C_1(s) = -\frac{\sqrt{\pi}}{2i\delta}\, s^2 e^{\frac{1}{4}(1 + \frac{4}{\delta R_e})s^2} \qquad -\pi < \arg s < 2\pi \tag{124}$$

Performing the inverse Fourier transform to (120), we obtain the required solution as an inverse Fourier integral

$$u_1(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_1(x,s)\, e^{is\tau}\, ds \tag{125}$$

It is difficult to evaluate the complete integral; however, the asymptotic behavior for large x and limiting behavior for small x are

studied.

Asymptotic Behavoir for Large x :

Using the asymptotic forms of the Hankel functions in $\hat{u}_1(x,s)$ in

(125) and evaluating the resulting integral, we obtain the asymptotic

solution for large x

$$u_1(x,\tau) \underset{x \to \infty}{\sim} \frac{1}{\delta x \left(1+\frac{4}{\delta R_e}\right)^{3/2}} \left[(\tau+x) e^{-\frac{(\tau+x)^2}{1+\frac{4}{\delta R_e}}} + (\tau-x) e^{-\frac{(\tau-x)^2}{1+\frac{4}{\delta R_e}}} \right]$$

$$+ O\left(\frac{1}{\delta x^2 \left(1+\frac{4}{\delta R_e}\right)^{3/2}}\right) \tag{126}$$

In the limit $x \to \infty$, $\frac{\tau}{x} \to -1$, the second term is exponentially

small and the solution matches the outer boundary condition (119). In

the limit $x \to \infty$, $\frac{\tau}{x} \to +1$, the first term is negligible and the

solution (126) predicts a reflected outgoing pulse.

Limiting Solution for Small x :

Replacing the Hankel functions in $\hat{u}_1(x,s)$ in (125) by their limit-

ing forms (121) and performing the integration on the resulting integral,

we obtain

$$u_1(x,\tau) \underset{x \to 0}{=} -\frac{1}{3} \frac{x}{\delta} \frac{1}{\left(1+\frac{4}{\delta R_e}\right)^{1/2}} \frac{d^3}{d\tau^3} \left[exp\left(-\frac{\tau^2}{1+\frac{4}{\delta R_e}}\right) \right] + O\left(\frac{x^3}{\delta \left(1+\frac{4}{\delta R_e}\right)^{1/2}}\right)$$
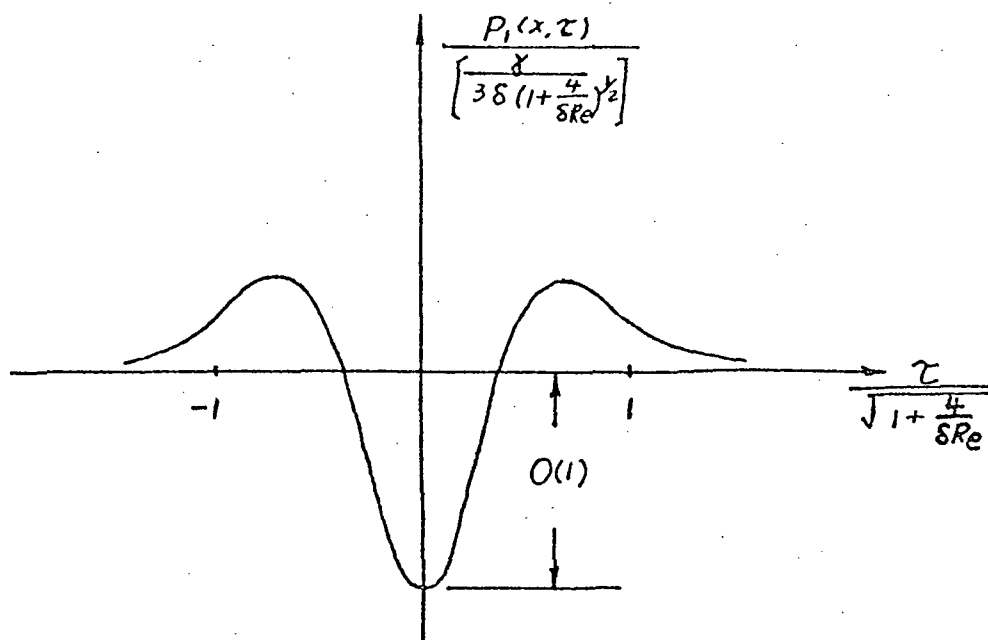
$$\tag{127}$$

Fig. 5 Pressure perturbation near and at the center (Eq.(129))

The pressure and density perturbations as $x \to 0$ are calculated from (92); they are

$$\underset{x \to 0}{P_1(x,\tau)} = \frac{1}{3}\frac{1}{\delta}\frac{1}{\left(1+\frac{4}{\delta Re}\right)^{1/2}} \frac{d^2}{d\tau^2}\left[\exp\left(-\frac{\tau^2}{1+\frac{4}{\delta Re}}\right)\right] + O\left(\frac{x^2}{\delta\left(1+\frac{4}{\delta Re}\right)^{1/2}}\right) \tag{128}$$

$$\underset{x \to 0}{p_1(x,\tau)} = \gamma\underset{x \to 0}{P_1(x,\tau)} \tag{129}$$

If we apply the analysis above to the problem posed by Eq.(91) with the other outer boundary condition (96), we have the following results:

Asymptotic Solution for Large x :

$$\underset{x \to \infty}{u_1(x,\tau)} \sim \frac{1}{\delta x}F(\tau+x) + \frac{1}{\delta x}F(\tau-x) + O\left(\frac{1}{\delta x^2}\right) \tag{130}$$

Limiting Solutions for Small x :

$$u_1(x,\tau) \underset{x \to 0}{=} \frac{+\sqrt{\pi}}{2\,\Gamma(5/2)} \frac{x}{\delta} \frac{d^2}{d\tau^2} F(\tau) + O\left(\frac{x^3}{\delta}\right) \qquad (131)$$

$$\rho_1(x,\tau) \underset{x \to 0}{=} \frac{-\sqrt{\pi}}{2\,\Gamma(5/2)} \frac{1}{\delta} \frac{d}{d\tau} F(\tau) + O\left(\frac{x^2}{\delta}\right) \qquad (132)$$

$$P_1(x,\tau) \underset{x \to 0}{=} \gamma\, \rho_1(x,\tau) \underset{x \to 0}{} \qquad (133)$$

Here the function $F(z)$ has the form

$$F(z) = z\left[\overline{\Phi}\left(\frac{\sqrt{\delta Re}\,(z+1)}{\sqrt{2}}\right) - \overline{\Phi}\left(\frac{\sqrt{\delta Re}\,(z-1)}{\sqrt{2}}\right)\right]$$

$$+ \frac{1}{\sqrt{\pi}\sqrt{\delta Re}}\left(e^{-\delta Re\left(\frac{z+1}{2}\right)^2} - e^{-\delta Re\left(\frac{z-1}{2}\right)^2}\right) \qquad (134)$$
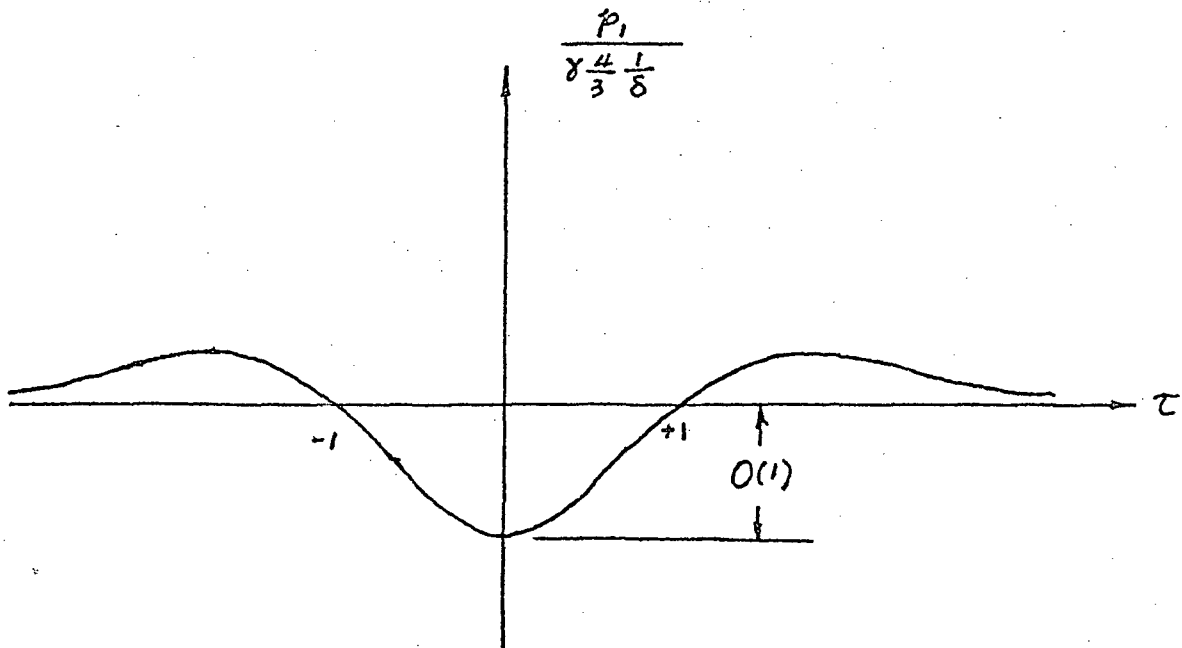

Fig. 6   Pressure perturbation near and at the center   (Eq.(133))

As in the cylindrical case, the velocity perturbation u_1 is linearly proportional to x as $x \rightarrow 0$ and the pressure and density perturbations p_1, ρ_1 are independent of x as $x \rightarrow 0$. The amplitude of the perturbations remains small, i.e., $O(\frac{1}{\delta}) \ll O(\frac{1}{\epsilon})$, throughout the flow field including the center of symmetry. However, the spherical and cylindrical cases differ in one important respect: that the reflected spherical pulse changes phase by 180-degree but the reflected cylindrical pulse does not, as indicated by the asymptotic solution in the cylindrical case (107) and the asymptotic solution in the spherical case (126). With the same incoming pulse, i.e., compressions ($p > 0$) followed by rarefactions ($p < 0$), the reflected spherical pulse has the profile with the rarefactions followed by the compressions, but the reflected cylindrical pulse has the same profile as the incoming pulse. Chernous'ko[9] used different method to solve the converging spherical compression or expansion wave (a jump discontinity) in a linear acoustic approximation; the result shows that a compression wave reflects as an expansion wave (and conversely) after which the undisturbed state is re-established.

The occurrence of 180-degree phase shift in spherical focusing but not in cylindrical focusing is due to the fact that spherical focusing is much stronger than cylindrical focusing. In the problem of bursting of a spherical ballon, an inward-propagating expansion wavefront is followed immediately by a compression wave. As the expansion wavefront approaches the center of symmetry, a portion of the compression wave coalesces with the expansion wavefront because of the strong focusing and a cusp is

formed at wavefront. The closer the wavefront approaches the center, the sharper the cusp becomes. As the wavefront reaches the center, there is no clear differentiation between the expansion wavefront and the portion of compression wave in the cusp; the flow particles at the center are affected simultaneously by the expansion and compression. We may say that the expansion wavefront is caught and replaced immediately by the compression wave right after the reflection. Thus, the 180-degree shift change occurs in the spherical focusing. On the other hand, the cylindrical focusing is weaker; the compression wave does not coalesce with the expansion wavefront and no cusp can be formed. Thus, no phase change occurs in this case.

Weak Nonlinear Case

In Chapter III, we have obtained the solutions for the converging pulse in the weak nonlinear case, i.e., Eq.(80) and Eq.(81). Both solutions describe an N-wave consisting of a front shock, an expansion zone, and a rear shock. As the N-wave approaches the interaction region $\eta \to \delta$, the amplitude of the N-wave becomes $O(\frac{\varepsilon}{\sqrt{\delta}})$ for $j = 1$ and $O(\frac{\varepsilon}{\delta\sqrt{\ln\frac{1}{\delta}}})$ for $j = 2$, which is still less than $O(1)$ since we have assumed $\frac{\varepsilon}{\delta} = O(1)$ in the weak nonlinear case. The major difference between the weak nonlinear case and the linear case is that the shock waves develop in the pulse-form due to the nonlinear effect; hence, two length scales are contained in the pulse. In our example of an N-wave, there are two shock waves whose thickness is at least one order in ε^{*} less than the pulse-length of the N-wave, and the size of the expansion zone is com-

parable to the pulse-length of the N-wave; more precisely, the expansion zone is the pulse-length less the two shock thicknesses. The interaction region is defined to be the same order as the pulse-length of the N-wave, thus, the propagation of the front and rear shocks in the interaction region is a long-time process because the shock must travel a distance much greater than its own thickness, and the propagation of the expansion zone between the front and rear shocks in the interaction region may be considered as a short-time process since its size is comparable to the interaction region. The existence of different scales for different portions of the N-wave enable us to consider each portion separately. We will first study the front shock of the converging N-wave as it enters the interaction. The analysis will be in general similar to that of Chapman's[16] except that the initial shock structure and shock trajectory are naturally determined from matching to the solution in converging region.

For convenience we change the independent variables in Eq.(80) and Eq.(81) from (η, ζ^*) to (η, ζ) using the relation $\zeta^* = \frac{\delta}{\varepsilon \Gamma} \zeta$ We obtain

$j = 1$ (from Eq.80) :

$$u_{1c}(\eta, \zeta; \varepsilon^*) = \frac{\zeta}{2\sqrt{\eta} \frac{\varepsilon \Gamma}{\delta}\left(\frac{\delta}{\varepsilon \Gamma} + 2 - 2\sqrt{\eta}\right)} \left[\tanh\left(\frac{1}{4} \frac{\zeta + \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}(1-\sqrt{\eta})}}{\varepsilon^* \frac{\varepsilon \Gamma}{\delta}\sqrt{\eta}\sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}(1-\sqrt{\eta})}}\right) \right.$$

$$\left. - \tanh\left(-\frac{\zeta - \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}(1-\sqrt{\eta})}}{\varepsilon^* \frac{\varepsilon \Gamma}{\delta}\sqrt{\eta}\sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}(1-\sqrt{\eta})}}\right) \right] + O(\varepsilon^*)$$

$$(135)$$

$j = 2$ (from Eq.81) :

$$U_{1C}(\eta, \xi^*; \varepsilon^*) = \frac{\xi}{2\eta \left(1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta\right)}\left[\tanh\left(-\frac{\xi + \sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}}{\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\eta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}}\right)\right.$$

$$\left. -\tanh\left(-\frac{\xi - \sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}}{\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\eta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}}\right)\right] + O(\varepsilon^*)$$

(136)

As the N-wave approaches the interaction region i.e., as $\eta \to \delta$, the thickness of the front shock, $\mathcal{W}(\frac{\eta}{\delta})^{\frac{j}{2}}$, which is represented by the denominator of hyperbolic tangent function in Eq.(80) and Eq.(81) and nondimensionalized by pulse-length λ , becomes

$j = 1$:
$$\mathcal{W} = O(\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\sqrt{\delta}) \equiv O(\frac{\sqrt{\delta}}{\varepsilon Re})$$
(137)

$j = 2$:
$$\mathcal{W} = O(\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\delta\sqrt{\ln\frac{1}{\delta}}) \equiv O(\frac{\delta}{\varepsilon Re}\sqrt{\ln\frac{1}{\delta}})$$
(138)

and the strength of the front shock Δ as $\eta \to \delta$ is

$j = 1$:
$$\Delta = O(\frac{\varepsilon}{\sqrt{\delta}})$$
(139)

$j = 2$:
$$\Delta = O(\frac{\varepsilon}{\delta\sqrt{\ln\frac{1}{\delta}}})$$
(140)

In order to study the flow field inside the front shock of the N-wave, we introduce a coordinate system that moves with shock and magnifies the shock thickness

$$x = x$$
$$y = \frac{\tau + x + \Delta f(x)}{\mathcal{W}}$$
(141)

where $f(x)$ depends on the local shock strength and will be determined

later. The shock strength Δ is small at least initially; we expand the dependent variables in series of powers of

$$u(x, \tau; \Delta) = \Delta u_1(x, y) + \Delta^2 u_2(x, y) + \cdots$$

$$\rho(x, \tau; \Delta) = \Delta \rho_1(x, y) + \Delta^2 u_2(x, y) + \cdots$$

$$p(x, \tau; \Delta) = \Delta P_1(x, y) + \Delta^2 P_2(x, y) + \cdots$$

$$T(x, \tau; \Delta) = \Delta T_1(x, y) + \Delta^2 T_2(x, y) + \cdots$$

$$S(x, \tau; \Delta) = \Delta S_1(x, y) + \Delta^2 S_2(x, y) + \cdots \tag{142}$$

Substitution of the coordinate system (141) and the series expansions of dependent variables (142) into the basic equations (83) to (87) yield a sequence of approximate equations; the first and second approximate equations in the sequence are

$$0(1, \frac{\Delta}{w}) :$$

$$\frac{\partial \rho_1}{\partial y} + \frac{\partial u_1}{\partial y} = 0$$

$$\frac{\partial u_1}{\partial y} + \frac{1}{\gamma}\frac{\partial P_1}{\partial y} = 0$$

$$\frac{\partial S_1}{\partial y} = 0$$

$$P_1 = T_1 + \rho_1$$

$$P_1 = S_1 + \gamma \rho_1 \tag{143}$$

$O(\frac{\Delta^2}{\varpi}, \Delta)$:

$$\frac{\partial P_2}{\partial y} + f'(z)\frac{\partial u_1}{\partial y} + \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y} + u_1\frac{\partial P_1}{\partial y} = 0$$

$$\rho_1\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y} + u_1\frac{\partial u_1}{\partial y} + \frac{1}{\gamma}\frac{\partial P_2}{\partial y} + \frac{1}{\gamma}f'(z)\frac{\partial P_1}{\partial y} - \frac{\partial^2 u_1}{\partial y^2} = 0$$

$$\rho_1\frac{\partial S_1}{\partial y} + T_1\frac{\partial S_1}{\partial y} + \frac{\partial S_2}{\partial y} + u_1\frac{\partial S_1}{\partial y} - K\frac{\partial^2 T_1}{\partial y^2} = 0$$

$$P_2 - T_2 - \rho_2 - \rho_1 T_1 = 0$$

$$P_2 - S_2 - \gamma S_1\rho_1 - \gamma\rho_2 - \frac{S_1^2}{2} - \frac{\gamma(\gamma-1)}{2}\rho_1^2 = 0$$

$$(144)$$

The first-order equations (143) are integrated with the application of the conditions that disturbances vanish as $y \longrightarrow -\infty$; we have

$$\rho_1 + u_1 = 0$$

$$u_1 + \frac{P_1}{\gamma} = 0$$

$$S_1 = 0$$

$$P_1 - T_1 - \rho_1 = 0$$

$$P_1 - S_1 - \gamma\rho_1 = 0$$

$$(145)$$

Upon substitution of above relations (145) into the second-order approximate equations (144), we obtain a single equation for u_1

$$A\frac{\partial^2 u_1}{\partial y^2} + f'(z)\frac{\partial u_1}{\partial y} = \Gamma u_1\frac{\partial u_1}{\partial y}$$

$$(146)$$

where

$$A = \frac{1 + \frac{K}{\gamma}(\gamma-1)}{2} \qquad \Gamma = \frac{\gamma+1}{2}$$

The flow field inside the front shock is governed by Eq.(146) in which the nonlinear and viscous effects are included. It is noted that this equation is similar to the boundary-layer equation in the converging region. Eq.(146) can be integrated twice immediately. The constants of integration are zero because of the conditions that disturbances vanish at $y = -\infty$ and the half value of shock strength is assumed at $y = 0$. The solution of Eq.(146) thus obtained is

$$u_1(x,y) = \frac{f'(x)}{\Gamma} \left[1 + \tanh\left(-\frac{f'(x)}{2A} y\right)\right]$$

(147)

The shock strength is the asymptotic limit of $u_1(x,y)$ as $y \rightarrow \infty$,

$$u_1(x,\infty) = 2\frac{f'(x)}{\Gamma}$$

(148)

The function $f'(x)$ is determined by matching the present solution to the solution in the converging region. The strength of the front shock of the N-wave in converging region is obtained from the boundary-layer solution (67) for $j = 1$ and (74) for $j = 2$ as $\bar{\xi} \rightarrow \infty$,

$j = 1$:

$$u_1(\eta,\infty;\varepsilon^*) = \frac{-1}{\sqrt{\eta}\sqrt{1 + 2\frac{\varepsilon\Gamma}{\delta}(1-\sqrt{\eta})}}$$

(149)

$j = 2$:

$$u_1(\eta,\infty;\varepsilon^*) = \frac{-1}{\eta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\eta}}$$

(150)

Rewriting the expressions (149) and (150) in terms of the variables of the interaction region and expanding the resultant expressions for small δ, with $\frac{\varepsilon\Gamma}{\delta}$ & x fixed, we have

$$j = 1 : \qquad u_1(\eta, \infty ; \varepsilon^*)_{x,y} = \frac{-1}{\sqrt{\delta} \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}} \sqrt{x}} \qquad (151)$$

$$j = 2 : \qquad u_1(\eta, \infty ; \varepsilon^*)_{x,y} = \frac{-1}{\delta \sqrt{\ln \frac{1}{\delta}} \; x} \qquad (152)$$

Before carrying out the matching, we should notice that the solution $u_1(x,y)$ is the first term of a series expansion in terms of δ and the solution of converging region $u_1(\eta, \bar{\xi} ; \varepsilon^*)$ is the first term of a series expansion in terms of ε, thus, the matching requires

$$\Delta u_1(x, y) = \varepsilon \, u_1(\eta, \bar{\xi} ; \varepsilon^*) \qquad (153)$$

Substituting the expressions for $u_1(x,y)$, (148), and $u_1(\eta, \xi ; \varepsilon^*)$, (151), in (153), we have $f'(x)$ in the cylindrical case

$$j = 1 : \qquad f'(x) = \frac{-\Gamma}{\sqrt{x} \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}}} \qquad (154)$$

Substituting the expressions for $u_1(x,y)$, (148), and $u_1(\eta, \bar{\xi} ; \varepsilon^*)$, (152), in (153), we have $f'(x)$ in the spherical case

$$j = 2 : \qquad f'(x) = -\frac{\Gamma}{2x} \qquad (155)$$

Substituting the expressions for $f'(x)$ in (147), we have the solution for the front shock of the N-wave in the interaction region,

$$j = 1 : \qquad u_1(x,y) = \frac{1}{2\sqrt{x}\sqrt{1 + 2\frac{\varepsilon \eta}{\delta}}} \left[1 + \tanh\left(\frac{\Gamma y}{4A\sqrt{x}\sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}}}\right) \right] \qquad (156)$$

$$j = 2 : \qquad u_1(x,y) = \frac{1}{2x} \left[1 + \tanh\left(\frac{\Gamma y}{4Ax}\right) \right] \qquad (157)$$

Equation (156) and equation (157) both exhibit local Taylor's shock structure; the shock thickness varies with the radial space coordinate x in the following manner:

$$j = 1 : \qquad (\lambda \sqrt{x}\, \varpi) \sim \sqrt{x}\, \frac{\sqrt{\delta}}{\varepsilon Re}\, \lambda \tag{158}$$

$$j = 2 : \qquad (\lambda\, x\, \varpi) \sim \frac{x\, \delta}{\varepsilon Re}\sqrt{ln\tfrac{1}{\delta}}\, \lambda \tag{159}$$

The shock thickness is useful for estimating the magnitude of derivatives in the basic equations (83) to (87). As indicated in (143) and 144), all the terms included in the first- and second-approximate equations contain only derivatives with respect to y (the derivatives across the shock wave). Using the solutions (156) and (157) and the relations for the shock thickness (158) and (159), we can estimate the terms in Eq.(146). The smallest one is

$$j = 1 : \qquad O\left(\frac{\Delta^3 Re}{x^{3/2}}\right) \tag{160}$$

$$j = 2 : \qquad O\left(\frac{\Delta^2 \varepsilon Re}{x^3 \delta \sqrt{ln\tfrac{1}{\delta}}}\right) \tag{161}$$

The largest terms omitted from the equations (83) to (87) describing the flow field within the shock wave are equivalent to derivatives with respect to x (derivatives along the wave), and from the solution obtained these are

$$j = 1 : \qquad O\left(\frac{\Delta}{x^{5/2}}\right) \qquad\qquad (162)$$

$$j = 2 : \qquad O\left(\frac{\Delta}{x^2}\right) \qquad\qquad (163)$$

Recall that $\Delta = \frac{\varepsilon}{\sqrt{\delta}}$ for $j = 1$ and $\Delta = \frac{\varepsilon}{\delta\sqrt{\ln\frac{1}{\delta}}}$ for $j = 2$. From above relations, it is easily seen that the omitted terms can affect the issue only for

$$j = 1 : \qquad \frac{\Delta^3 Re}{x^{5/2}} \approx \frac{\Delta}{x^{5/2}} \qquad \therefore \quad \varepsilon Re \approx O(1)$$

$$j = 2 : \qquad \frac{\Delta^2 \varepsilon Re}{\delta\sqrt{\ln\frac{1}{\delta}}x^3} \approx \frac{\Delta}{x^2} \qquad \therefore \quad \frac{Re}{x} \approx O(1)$$

which contradict our assumption that $\frac{\varepsilon}{\delta} = O(1)$, $\varepsilon Re \gg 1$. Another phenomenon which may affect the current result is the reflected waves from the origin. However, reflection starts only when the shock wave propagates to a distance from the origin which is comparable to the width of the blurred zone of the shock. For the cylindrical case this distance is $x = O\left(\frac{1}{\varepsilon Re^2}\right)$, and the shock strength at this distance is $O(\varepsilon Re)$ which is much greater than 1 according to the assumption $\delta Re \gg 1$. For the spherical case the shock thickness is linearly proportional to x (the position of shock), therefore, it is always a factor $\frac{\delta\sqrt{\ln\frac{1}{\delta}}}{\varepsilon Re}$ less than the distance x, as obviously seen from (159), regardless how small the x is; the shock strength always becomes large (as $x \rightarrow 0$) before any reflection starts to occur.

The above arguments lead to the conclusion that under the assumptions

$\varepsilon^* \ll 1$, $\frac{\varepsilon}{\delta} = 0(1)$ there is no small-pertubation solution for the shock waves in the vicinity of the axis or center of symmetry, even though the second-order nonlinear, viscous, and heat-conducting effects are taken into account. In the neighborhood of the origin where the shock strength becomes much greater than 1, the unperturbed basic equations (1) to (5) must be used.

Since the initial amplitude of the pulse characterized by ε is small, say, 10^{-3} , the size of the region within which the shock strength becomes very large, i.e., $0(\varepsilon Re)$ is expected to be small. The radius of this region can be estimated from the solution (156) (or (157)):

$$j = 1 : \qquad x_o = \left(\frac{1}{\varepsilon Re^2} \right) \tag{164}$$

$$j = 2 : \qquad x_o = \left(\frac{1}{\delta \sqrt{\ln \frac{1}{\delta}} \, Re} \right) \tag{165}$$

Within x_o the shock wave becomes very strong and the energy dissipation due to the entropy rise across the shock becomes significant. It is interesting to know the energy dissipation in this region after the shock passes through. To estimate it we use the similarity solution of strong-shock implosion given by Stanyukovich[12] (see also reference (11)) who solved the unperturbed basic equations (1) to (5) without the dissipation terms; the dissipative effects are included through the Rankine-Hugoniot relations. At the instant the converging shock reaches the origin, the limiting relationships between gas variables and space coordinate r from the similarity solutions are

$$u \sim a \sim D^{1/2} r^{-\frac{1-\alpha}{\alpha}}$$

$$p \sim T \sim D^{2/\alpha} r^{-\frac{2(1-\alpha)}{\alpha}}$$

$$\rho \sim \text{constant} \tag{166}$$

where u , a , p , T and ρ are dimensional gas variables; D is a dimensional parameter containing only units of length and time and α is called similarity exponent (a pure numerical number). For strong implosion in a perfect gas with $\gamma = \dfrac{C_p}{C_v} = 7/5$, the value of α is 0.834 in the cylindrical case and 0.717 in the spherical case. The total energy dissipation E_d in a region with radius r_0 is

$$E_d = \int_0^{r_0} T \Delta S \, z^j \pi r^j \rho \, dr = \int_0^{r_0} \pi \rho T C_v \, \log \frac{p}{\rho^\gamma} z^j r^j \, dr \tag{167}$$

Substituting the limiting relationships (166) and the value of α into (167) and performing the integration, we obtain

$j = 1$:

$$E_d \sim r_0^{1.6} \log r_0 \tag{168}$$

where

$$r_0 = R_0 \delta x_0 = \frac{R_0 \delta}{\varepsilon R_e^2}$$

$j = 2$:

$$E_d \sim r_0^{2.21} \log r_0 \tag{169}$$

where

$$r_0 = R_0 \delta x_0 = \frac{R_0}{R_e \sqrt{\ln \frac{1}{\delta}}}$$

The energy per unit volume for a small disturbance that is propagated within a gas initially at rest can be found in many texts (e.g. Zel'dovich and Raizer[26]). It is

$$\mathcal{E} = h_0 (\rho - \rho_0) + \rho_0 u^2$$

(170)

The first-order change in the above expression is related to the change in total volume of the gas that occurred as a result of the disturbance. If the disturbance was created in such a manner that the total volume remained unchanged, then the perturbation of the entire energy of the entire gas is a quantity of second order in $(\rho - \rho_0)$, since the term proportional to $(\rho - \rho_0)$ vanishes in the process of integration over the volume. Thus, for an N-wave initially occupying a cylindrical or spherical shell between $R_0 + \frac{\lambda}{2}$ and $R_0 - \frac{\lambda}{2}$, the total energy of the N-wave is

$$j = 1 : \quad E = \mathcal{E} \left[\pi (R_0 + \tfrac{\lambda}{2})^2 - \pi (R_0 - \tfrac{\lambda}{2})^2 \right] \approx 2\pi R_0 \rho_0 \, \mathcal{E}^2 \lambda$$

(171)

$$j = 2 : \quad E = \mathcal{E} \left[\tfrac{4}{3} \pi (R_0 + \tfrac{\lambda}{2})^3 - \tfrac{4}{3} \pi (R_0 - \tfrac{\lambda}{2})^3 \right] \approx 4\pi R_0^2 \rho_0 \, \mathcal{E}^2 \lambda$$

(172)

The ratio of the energy dissipation due to the entropy rise across the strong shock in the region defined by r_0 to the total energy of the N-wave is

$$j = 1 : \quad \frac{E_d}{E} \sim \frac{\left(\frac{R_0 \delta}{\mathcal{E} R_e^2}\right)^{1.6} \log \frac{R_0 \delta}{\mathcal{E} R_e^2}}{\mathcal{E}^2 R_0 \, \lambda} \sim \frac{\log R_e}{R_e^{3.2} \, \delta}$$

(173)

$$j = 2 : \qquad \frac{E_d}{E} \sim \frac{\left(\frac{R_0}{Re \sqrt{\ln\frac{1}{\delta}}}\right)^{2.21} \log \frac{R_0}{Re \sqrt{\ln\frac{1}{\delta}}}}{\varepsilon^2 R_0^2 \lambda} \sim \frac{\log Re}{\delta^3 (\ln\frac{1}{\delta})^{1.5} Re^{2.21}}$$

$$(174)$$

The above results show that the ratio of the amount of energy dissipated in the region where the shock becomes very strong to the total energy of the N-wave depends on the parameters ε , δ , and Re . For example, the convergence of an N-wave in air at standard conditions with $R_0 = 10^m$ ft , $\delta = \varepsilon = 10^{-n}$, gives

$$j = 1 : \qquad \frac{E_d}{E} \approx \frac{7 - n + m}{10^{22.4 - 4.2n + 3.2m}}$$

$$j = 2 : \qquad \frac{E_d}{E} \approx \frac{7 - n + m}{(n)^{1.05} \, 10^{15.5 - 5.21n + 2.21m}}$$

Thus, the energy dissipation is negligible in comparison with the total energy of the N-wave, if the following condition is satisfied:

$$j = 1 : \qquad k_1 = 22.4 - 4.2n + 3.2m > 1 \qquad (173\ a)$$

$$j = 2 : \qquad k_2 = 15.5 - 5.21n + 2.21m > 1 \qquad (174\ a)$$

It is easy to see that the condition for the spherical case is more restrictive than that for the cylindrical case. Typically, a spark generated N-wave might have $m = 1$, $n = 3$, which gives $k_1 = 13$, $k_2 = 2.1$.

This completes our analysis of the shocks of the N-wave in the interaction region. We proceed to examine the flow field between the front and

rear shocks of the N-wave, i.e., the expansion zone. As already men-
tioned, the width of the expansion zone is comparable to the pulse-length
of the N-wave. One important difference between the flow field in the
expansion zone and that in shock waves is that the gas variables change
smoothly i.e., the gradients of gas variables are not large in the expan-
sion zone. As the front shock converges to the origin the strength of
this shock tends to become singular. In addition, the shock structure
as well as a small segment of the expansion zone behind the shock front
exhibit very large gradients and large disturbances! However, following
the above described portion of the wave, we have the remainder of the
N-wave, which consists of the remaining portion of the expansion region
as well as the rear shock, where disturbances are still small. To study
the flow field in the expansion zone between the front and rear shocks,
we assume the relations

$$\frac{\partial}{\partial x} \sim \frac{\partial}{\partial x^2} \sim \frac{\partial}{\partial \tau} \sim O(1) \qquad (175)$$

with the understanding that these relations are violated locally in the
front and rear shocks as they approach the origin. The amplitude of the
N-wave is $O(\Delta)$ initially. Hence, we expand the dependent variables
u , ρ , p , T and S in power series of Δ as shown in (142).
Upon substitution of these expansions into the basic equations (84) to
(87) and using the relations (175), we find that the first-order approxi-
mate equations can be reduced to

$$\frac{\partial^2 u_i}{\partial \tau^2} = \frac{\partial^2 u_i}{\partial x^2} + \frac{j}{x}\left(\frac{\partial u_i}{\partial x} - \frac{u_i}{x^2}\right) \tag{176}$$

$$\frac{\partial u_i}{\partial \tau} + \frac{\partial p_i}{\partial x} = 0$$

$$p_i = \gamma \rho_i$$

$$S_i = 0 \tag{177}$$

Therefore, the governing equation of the flow field in the expansion zone of the N-wave is, to a first-order approximation, linear and inviscid wave equation. It should be remembered that this equation doesn't apply to the flow field inside the shock waves, since the condition (175) is locally violated inside the shocks. Furthermore, Eq.(176) and Eq.(177) are meaningful only if the conditions (173 a) and (174 a) are satisfied, i.e., the energy loss due to entropy rise across the shock is very small in comparison with the total energy of the N-wave. We will first solve Eq.(176) for the cylindrical case.

(a) $j = 1$:

The boundary conditions for Eq.(176) are that the solution should vanish at the axis of symmetry for all time and it should match the solution of the converging region as $x \to \infty$, $\frac{\tau}{x} \longrightarrow -1$. The outer boundary condition is derived from Eq.(135). It is well known that in linear and inviscid approximation the position of front and rear shocks relative to the central point of the N-wave and shock thickness stay constant; only the amplitude will vary because of focusing. Thus, Eq.(135) is modified by approximating the shock thickness and shock positions by their values at $\eta = \delta$, i.e., at the boundary of the interaction region. Setting

$\eta = \delta$ in the argument of the hyperbolic tangent functions in Eq. (135), changing the solution into the variables of interaction region (x, τ), and expanding the resulting expression for small δ with τ , x , fixed, we obtain the matching condition for Eq. (176)

$$u_1(x,\tau) \underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{\sim} \frac{\tau + x}{2\sqrt{x}\,(1+2\frac{\delta}{\varepsilon\Gamma})} \left[\tanh\left(-\frac{1}{4} \frac{\tau + x - \sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}{\varepsilon^a \frac{\varepsilon\Gamma}{\delta} \sqrt{\delta} \sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}\right) \right.$$

$$\left. - \tanh\left(-\frac{1}{4} \frac{\tau + x + \sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}{\varepsilon^* \frac{\varepsilon\Gamma}{\delta} \sqrt{\delta} \sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}\right) \right]$$

$$(178)$$

The inner boundary condition is

$$u(x, \tau) = 0 \qquad\qquad \text{at } x = 0 \text{ for all } \tau \text{ .} \qquad (179)$$

The procedures used to solve Eq. (177) with the boundary conditions (178) and (179) are the same as those used for the linear cylindrical case. Application of the Fourier transform to Eq. (177) and boundary conditions (178) and (179) yields

$$\frac{\partial^2 \hat{u}_1}{\partial x^2} + \frac{2}{x} \frac{\partial \hat{u}_1}{\partial x} + (s^2 - \frac{1}{x}) \hat{u}_1 = 0$$

$$(180)$$

and the boundary conditions

$$\hat{u}_1(x,s) \underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{\sim} \frac{1}{2(1+2\frac{\varepsilon\Gamma}{\delta})\sqrt{x}} e^{isx} \int_{-\infty}^{\infty} P(\xi) e^{-is\xi} d\xi$$

$$(181)$$

$$\hat{u}_1(x,s) = 0 \qquad \text{at } x = 0 \qquad\qquad (182)$$

where

$$P(\xi) = \xi\left[\tanh\left(\frac{\xi + \sqrt{1+2\frac{\epsilon\Omega}{\delta}}}{4\epsilon^*\frac{\epsilon\Omega}{\delta}\sqrt{\delta}\sqrt{1+2\frac{\epsilon\Omega}{\delta}}}\right) - \tanh\left(\frac{\xi - \sqrt{1+2\frac{\epsilon\Omega}{\delta}}}{4\epsilon^*\frac{\epsilon\Omega}{\delta}\sqrt{\delta}\sqrt{1+2\frac{\epsilon\Omega}{\delta}}}\right)\right]$$

(183)

$$Q(s) = \int_{-\infty}^{\infty} P(\xi)\, e^{-is\xi}\, d\xi$$

(184)

The general solution of Eq. (180) is

$$\hat{u}_1(x,s) = C_1(s)\, H_1^{(1)}(xs) + C_2(s)\, H_1^{(2)}(xs)$$

(185)

The functions $C_1(s)$ and $C_2(s)$ are determined from the boundary conditions (181) and (182) as well as the properties of Hankel functions (102) and (103). The results are

$$C_1(s) = \sqrt{\frac{\pi}{2}}\, e^{\frac{3}{4}\pi i}\, \frac{1}{2\left(1+2\frac{\epsilon\Omega}{\delta}\right)}\, \sqrt{s}\, Q(s) \qquad -\pi < \arg s < 2\pi$$

(186)

$$C_2(s) = C_1(s)$$

(187)

The solution in the physical plane (x, τ) is obtained by using the inverse Fourier transform to (185) in which $C_1(s)$ and $C_2(s)$ are given by (186) and (187)

$$u_1(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_1(x,\tau)\, e^{is\tau}\, ds$$

(188)

Asymptotic Behavior for Large x :

The asymptotic forms of Hankel function (103) are used in $u_1(x,s)$ in (188), and evaluation of the resultant integral yields the asymptotic solution for large x

$$u_1(x,\tau) \underset{x \to \infty}{\sim} \frac{1}{2(1+2\frac{\varepsilon\Gamma}{\delta})\sqrt{x}} \left\{ (\tau+x)\left[\tanh\left(\frac{\tau+x+\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}{4\,\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\sqrt{\delta}\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}} \right) \right. \right.$$

$$\left. - \tanh\left(\frac{\tau+x-\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}{4\,\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\sqrt{\delta}\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}} \right) \right] - (\tau-x)\left[\tanh\left(\frac{\tau-x+\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}{4\,\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\sqrt{\delta}\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}} \right) \right.$$

$$\left. \left. - \tanh\left(\frac{\tau-x-\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}}{4\,\varepsilon^*\frac{\varepsilon\Gamma}{\delta}\sqrt{\delta}\sqrt{1+2\frac{\varepsilon\Gamma}{\delta}}} \right) \right] \right\} + O\left(\frac{1}{x^{3/2}(1+2\frac{\varepsilon\Gamma}{\delta})} \right)$$

$$(189)$$

It is easily seen that as $x \to \infty$, $\frac{\tau}{x} \to -1$, the last two terms are exponentially small; the solution (189) matches the outer boundary condition (178). As $x \to \infty$, $\frac{\tau}{x} \to +1$, the first two terms are exponentially small and the solution predicts a reflected N-wave which has the same profile as the incoming N-wave. The accuracy of this predictions is justified in the cylindrical case since the energy dissipation is negligible in comparison with the total energy of the N-wave.

Limiting Solution for Small x :

Using the limiting relationships of Hankel function (103) for $\hat{u}_1(x,s)$ in (188), we have

$$u_1(x,\tau)\underset{x \to 0}{} = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_1(s)\,2\,J_1(xs)\underset{x \to 0}{}\,e^{is\tau}\,ds$$

$$= \frac{x}{(1+2\frac{\varepsilon\Gamma}{\delta})}\sqrt{\frac{\pi}{2}}\,e^{3/4\pi i}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} s^{3/2}Q(s)\,e^{is\tau}ds + O\left(\frac{x^3}{1+2\frac{\varepsilon\Gamma}{\delta}} \right)$$

$$= \frac{x}{2\sqrt{2}\,(1+2\frac{\varepsilon\rho}{\delta})}\,\frac{d^2}{d\tau^2}\int_{-\infty}^{\tau}\frac{P(\xi)}{\sqrt{\tau-\xi}}\,d\xi \;+\; O\left(\frac{x^3}{1+2\frac{\varepsilon\rho}{\delta}}\right)$$

$$(190)$$

where $P(\xi)$ is given by (183).

The pressure and density perturbations are calculated from (177) and (190); the results are

$$\rho_1(x,\tau)\Big|_{x\to 0} \;=\; \frac{-1}{2\sqrt{2}\,(1+2\frac{\varepsilon\rho}{\delta})}\,\frac{d}{d\tau}\int_{-\infty}^{\tau}\frac{P(\xi)}{\sqrt{\tau-\xi}}\,d\xi \;+\; O\left(\frac{x^2}{1+2\frac{\varepsilon\rho}{\delta}}\right)$$

$$(191)$$

$$P_1(x,\tau)\Big|_{x\to 0} \;=\; \gamma\,\rho_1(x,\tau)\Big|_{x\to 0}$$

$$(192)$$

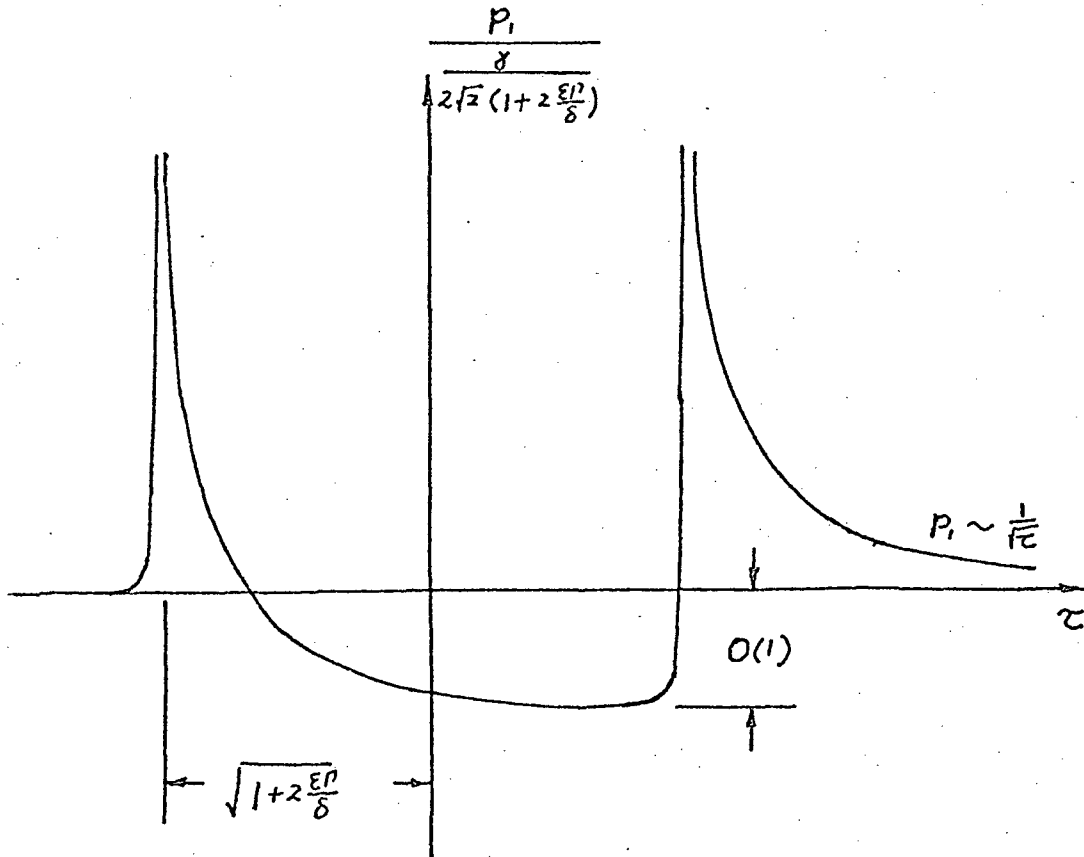where $P(\xi)$ is given by (183).

Fig. 7   Pressure perturbation near and at the axis   (Eq.(192))

The above limiting solutions give qualitative description of the disturbances at the focus. The amplitude evaluated at the instant when the front or rear shock reaches the focus is much greater than 1 . However, the flow field between the front and rear shock waves remains that of small perturbation; i.e., the amplitude is of $O(\triangle)$.

(b) $j = 2$ (spherical case) :

The boundary conditions for Eq.(176) are that the solution should vanish at the center of symmetry for all time and it should match the solution of the converging region as $x \to \infty$, $\dfrac{\tau}{x} \to -1$. The outer boundary condition is derived from Eq.(136) by setting $\eta = \delta$ in the argument of the hyperbolic tangent functions and changing the solution into the variables of interaction region (x, τ) , and expanding the resulting expression for small δ with τ , x , $\dfrac{\varepsilon}{\delta}$ fixed, we have

$$
\underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{u_1(x,\tau)} \sim \frac{\tau + x}{2x} \left[\tanh\left(\frac{\tau + x + \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln \delta}}{4 \varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln \delta}} \right) \right.
$$

$$
\left. - \tanh\left(\frac{\tau + x - \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln \delta}}{4 \varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln \delta}} \right) \right]
$$

The inner boundary condition is

$$
u_1(x, \tau) = 0 \qquad\qquad \text{at} \quad x = 0 \quad \text{for all} \quad \tau
$$

Applying the Fourier transform to Eq.(176) with $j = 1$ and the above boundary conditions, we have

$$\frac{\partial^2 \hat{u}_1}{\partial x^2} + \frac{2}{x}\frac{\partial \hat{u}_1}{\partial x} + (s^2 - \frac{2}{x^2})\hat{u}_1 = 0$$

with the boundary conditions

$$\hat{u}_1(x,s) \underset{\substack{x \to \infty \\ \frac{\tau}{x} \to -1}}{\sim} \frac{1}{2x} e^{isx} \int_{-\infty}^{\infty} M(\xi) e^{is\xi} d\xi$$

$$= \frac{1}{2x} e^{isx} N(s)$$

where

$$M(\xi) = \xi \left[\tanh\left(\frac{\xi + \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln\delta}}{4\varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln\delta}}\right) - \tanh\left(\frac{\xi - \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln\delta}}{4\varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 - \frac{\varepsilon \Gamma}{\delta}\ln\delta}}\right) \right]$$

$$N(s) = \int_{-\infty}^{\infty} M(\xi) e^{-is\xi} d\xi$$

The solution in the transformed plane is

$$\hat{u}_1(x,s) = -\frac{s}{2} Q(s) \sqrt{\frac{\pi}{2\pi s}} \left[H_{3/2}^{(1)}(xs) + H_{3/2}^{(2)}(xs) \right]$$

Performing the inverse Fourier transform, we obtain the solution in the physical plane (x, τ)

$$u_1(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_1(x,\tau) e^{is\tau} ds$$

The asymptotic behavior as $x \to \infty$ and limiting behavior as $x \to 0$ of the inverse Fourier integral are given below.

Asymptotic Behavior for Large x :

$$u_1(x,\tau) \underset{x \to \infty}{\sim} \frac{1}{2x} \left[M(\tau+x) - M(\tau-x) \right]$$

Limiting Solution for Small x :

$$u_1(x,\tau) \underset{x \to 0}{=} \frac{+\sqrt{\pi}\, x}{4\,\Gamma(5/4)} \quad \frac{d^2}{d\tau^2} M(\tau)$$

$$\rho_1(x,\tau) \underset{x \to 0}{=} \frac{-3\sqrt{\pi}}{4\,\Gamma(5/4)} \quad \frac{d}{d\tau} M(\tau)$$

$$P_1(x,\tau) \underset{x \to 0}{=} \gamma \rho_1(x,\tau) \underset{x \to 0}{}$$

where
$$M(\tau) = \tau \left[\tanh\left(\frac{\tau + \sqrt{1 - \frac{\varepsilon \Gamma}{\delta} \ln \delta}}{4\,\varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 - \frac{\varepsilon \Gamma}{\delta} \ln \delta}} \right) - \tanh\left(\frac{\tau - \sqrt{1 - \frac{\varepsilon \Gamma}{\delta} \ln \delta}}{4\,\varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 - \frac{\varepsilon \Gamma}{\delta} \ln \delta}} \right) \right]$$
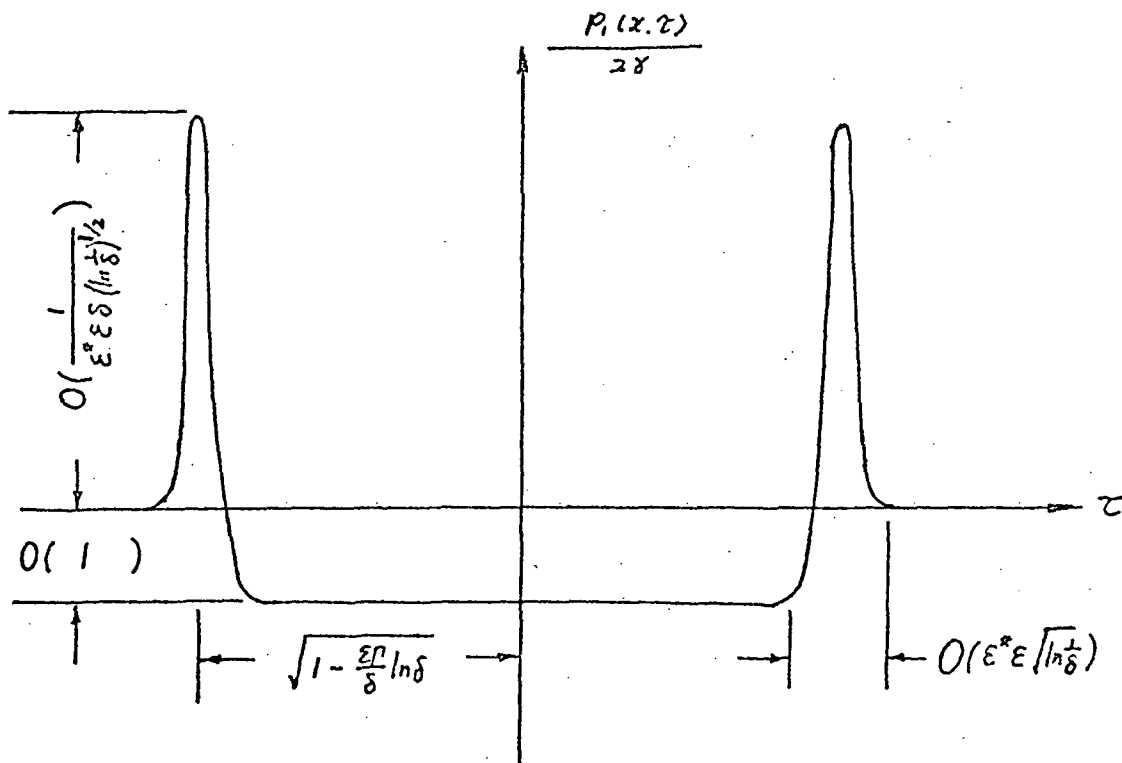

Fig. 8 Pressure perturbation near and at the center

V. DIVERGING REGION

The outgoing diverging pulses are briefly discussed in this Chapter. The study of the diverging pulse is very similar to that of the converging pulse except for some trivial change in the anlysis. In Chapter IV, we have obtained the solution of interaction region which predicts the outgoing pulse. The asymptotic form of this solution as $x \rightarrow \infty$ and $\frac{\tau}{x} \rightarrow +1$ provides the initial pulse form for the diverging pulse. The initial pulse-form derived from the asymptotic solution of the interaction region is prescribed at $r = r_o$, $r_o = O(\lambda)$. We wish to find the motion of the pulse at large r , i.e., $r \gg r_o$. The basic equations are Eqs.(1) to (5) from which the governing equation for the diverging pulse is to be derived.

To nondimensionalize the system of equations (1) to (5), we use the dimensionless variables given by (6) and define the dimensionless independent variables by

$$r' = \frac{r}{R_o - r_o} \qquad\qquad t' = \frac{a_o t}{R_o - r_o} \qquad (193)$$

Here R_o is same radius $\gg r_o$.

If relations (193) and (6) are substituted in the basic equations (1) to (5) and all the primes are dropped afterwards, the nondimensional basic equations thus obtained are exactly the same as those given by (8) to (12) . The only modification in the equations is the parameter Re_{R_o} in which R_o is replaced by $R_o - r_o$. We introduce the following coordinate system to study the diverging pulse:

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$$\eta = r \qquad\qquad \xi = \frac{t + r}{\delta} \qquad\qquad (194)$$

where $\delta = \frac{\lambda}{R_0 - \lambda} \approx \frac{\lambda}{R_0}$ is assumed much less than 1 .

The initial amplitude of the diverging pulse at $\eta = \eta_0 = \frac{r_0}{R_0 - r_0}$

is known to be small; hence we can expand the gas variables in terms

of powers of ε as given by (17) . Substitution of relations (17) and

(194) into the system of nondimensional equations (8) to (12) yields a

sequence of approximate equations; the first-order and second-order

equations can be reduced to a single equation for u_1 .

$$\frac{\partial u_1}{\partial \eta} + \frac{j}{2}\frac{u_1}{\eta} - \frac{\varepsilon \Gamma}{\delta} u_1 \frac{\partial u_1}{\partial \xi} = \frac{1}{\delta Re}\frac{\partial^2 u_1}{\partial \xi^2} \qquad\qquad (195)$$

Eq.(195) is the governing equation for the outgoing diverging pulses.

It is noted that the equation is identical to Eq.(21) except for the sign

difference in front of the term on the right-hand side of the equation.

Therefore, all the techniques used before may be applied here. In the

following analysis we omit the detailed calculations and present the final

results only.

Linear Case

If the initial amplitude of the pulse characterized by ε is much

less than δ , the nonlinear term $\frac{\varepsilon \Gamma}{\delta} u_1 \frac{\partial u_1}{\partial \xi}$ in Eq.(195) can be

neglected and Eq.(195) becomes

$$\frac{\partial u_1}{\partial \eta} + \frac{j}{2}\frac{u_1}{\eta} = \frac{1}{\delta Re}\frac{\partial^2 u_1}{\partial \xi^2} \qquad\qquad (196)$$

The initial condition of the equation is obtained from the asymptotic solutions (107) & (126) by taking the limit $x \to \infty$, $\frac{\tau}{x} \to +1$ and expressing them in the present variables (η, ξ). The final form is

$$u_1(\eta, \xi) = \frac{(-1)^j}{\eta^{1/2}} \frac{1}{\left(1 + \frac{4}{\delta Re}\right)^{3/2}} \xi \, e^{-\frac{\xi^2}{1 + \frac{4}{\delta Re}}} \qquad \text{at} \qquad \eta = \eta_0 \tag{197}$$

The solution of Eq.(196) with initial condition (197) is obtained by the same method as used for Eq.(23). The solution is

$$u_1(\eta, \xi) = \frac{(-1)^j}{\eta^{1/2}} \frac{1}{1 + \frac{4}{\delta Re} + \frac{4(\eta - \eta_0)}{\delta Re}} \xi \, e^{-\frac{\xi^2}{1 + \frac{4}{\delta Re} + \frac{4(\eta - \eta_0)}{\delta Re}}} \tag{198}$$

Solution (198) shows that the amplitude decays and pulse-length increases as the pulse diverges (η increases) .

Weak Nonlinear Case

   If the initial amplitude of the diverging pulse is large such that $\frac{\varepsilon \Gamma}{\delta} = O(1)$, the complete equation (195) must be used. We first consider the cylindrical case.

(a) $j = 1$ (cylindrical case) :

   Setting $j = 1$ in Eq.(195) and changing the variable ξ to ξ^* , we have

$$\frac{\partial u_1}{\partial \eta} + \frac{1}{2} \frac{u_1}{\eta} - u_1 \frac{\partial u_1}{\partial \zeta^*} = \varepsilon^* \frac{\partial^2 u_1}{\partial \zeta^{*2}} \tag{199}$$

where $\zeta^* = \frac{\delta}{\varepsilon \Gamma} \zeta$ and $\varepsilon^* = \frac{\delta}{\varepsilon^2 Re \, \Gamma}$.

The initial condition for Eq. (199) is derived from (189) in which the limiting process $x \to \infty$, $\frac{\tau}{x} \to +1$ is taken. Writing the resultant form in η , ζ^* variables, we have the initial condition as follows

$$u_1(\eta, \zeta^*) = \frac{\frac{\varepsilon \Gamma}{\delta} \zeta^*}{2\sqrt{\eta_0}(1 + \frac{\varepsilon \Gamma}{\delta})} \left[\tanh\left(\frac{\frac{\varepsilon \Gamma}{\delta} \zeta^* - \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}}}{4 \varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}}} \right) \right.$$

$$\left. - \tanh\left(\frac{\frac{\varepsilon \Gamma}{\delta} \zeta^* + \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}}}{4 \varepsilon^* \frac{\varepsilon \Gamma}{\delta} \delta \sqrt{1 + 2\frac{\varepsilon \Gamma}{\delta}}} \right) \right] \quad \text{at} \quad \eta = \eta_0 \tag{200}$$

Equation (199) with initial condition (200) can be solved by the method of matched asymptotic expansions. The procedure of solving this problem is exactly the same as those used in Chapter III for the weak nonlinear case. Thus, only the final results are given here.

(i) Outer Solution

$$u_1(\eta, \zeta^*; \varepsilon^*) = \begin{cases} \dfrac{-\zeta^*}{\sqrt{\eta}\,(2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon \Gamma}(1 + 2\frac{\varepsilon \Gamma}{\delta}))} + O(\varepsilon^*) & |\zeta^*| \le \zeta_m^* \\[4mm] 0 & |\zeta^*| > \zeta_m^* \end{cases} \tag{201}$$

where $\zeta_m^* = \sqrt{\frac{\delta}{\varepsilon \Gamma}} \sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon \Gamma}(1 + 2\frac{\varepsilon \Gamma}{\delta})}$

(ii) Boundary-Layer Solutions

The boundary-layer solution at $\zeta^* = -\sqrt{\frac{\delta}{\varepsilon \Gamma}} \sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon \Gamma}(1 + 2\frac{\varepsilon \Gamma}{\delta})}$:

$$u_1(\eta, \bar{\xi}; \varepsilon^*) = \frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\sqrt{\eta}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}} \left[1 + \tanh\left(\frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}\,\bar{\xi}}{4\sqrt{\eta}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}\right)\right]$$

$$+ O(\varepsilon^*) \qquad (202)$$

where $\quad \bar{\xi} = \dfrac{\xi^* + \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}{\varepsilon^*}$

The boundary-layer solution at $\quad \xi^* = +\sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}$:

$$u_1(\eta, \hat{\xi}; \varepsilon^*) = \frac{-\sqrt{\frac{\delta}{\varepsilon\Gamma}}}{2\sqrt{\eta}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}} \left[1 - \tanh\left(\frac{\sqrt{\frac{\delta}{\varepsilon\Gamma}}\,\hat{\xi}}{4\sqrt{\eta}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}\right)\right]$$

$$+ O(\varepsilon^*)$$

$$(203)$$

where $\quad \hat{\xi} = \dfrac{\xi^* - \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}{\varepsilon^*}$

(iii) Uniformly Valid Composite Solution

$$u_1(\eta, \xi^*; \varepsilon^*) = \frac{\xi^*}{2\sqrt{\eta}\left(2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})\right)} \left[\tanh\left(\frac{\xi^* - \frac{\delta}{\varepsilon\Gamma}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}{4\varepsilon^*\sqrt{\eta}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}\right) \right.$$

$$\left. - \tanh\left(\frac{\xi^* + \frac{\delta}{\varepsilon\Gamma}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}{4\varepsilon^*\sqrt{\eta}\sqrt{2\sqrt{\eta} - 2\sqrt{\eta_0} + \frac{\delta}{\varepsilon\Gamma}(1+2\frac{\varepsilon\Gamma}{\delta})}}\right)\right] + O(\varepsilon^*)$$

$$(204)$$

Solution (204) shows that the pulse-length and shock thickness increase but the amplitude decreases as the pulse diverges. This solution is not valid for very large η , since then the shock thickness becomes comparable to the pulse-length as indicated by Seebass[27].

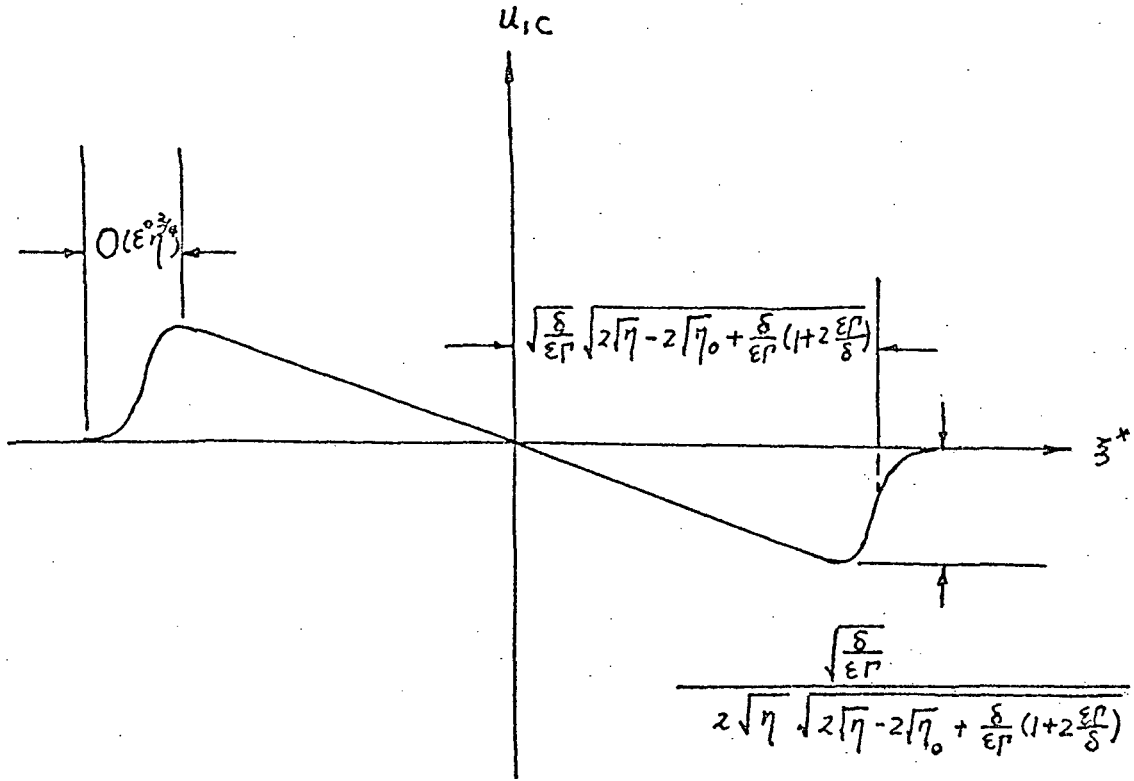Fig. 9 Velocity profile of the diverging pulse (Eq.(204))

(b) $j = 2$ (spherical case) :

Setting $j = 2$ in Eq.(195) and changing the variable ξ to ξ^* ,

we have

$$\frac{\partial u_I}{\partial \eta} + \frac{u_I}{\eta} - u_I \frac{\partial u_I}{\partial \xi^*} = \varepsilon^* \frac{\partial^2 u_I}{\partial \xi^{*2}}$$

The initial condition is derived from the asymptotic solution for large

x and $\dfrac{\tau}{x} \rightarrow +1$ of the interaction region in last chapter; the final form

in η , ξ^* variables is

$$u_1(\eta, \zeta^*) = \frac{\zeta^*}{2\eta_0 \frac{\delta}{\varepsilon\Gamma}(1 - \frac{\varepsilon\Gamma}{\delta}\ln\delta)} \left[\tanh\left(\frac{\zeta^* + \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\delta}}{4\varepsilon^*\delta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\delta}}\right) \right.$$

$$\left. - \tanh\left(\frac{\zeta^* - \sqrt{\frac{\delta}{\varepsilon\Gamma}}\sqrt{\frac{\delta}{\varepsilon\Gamma} - \ln\delta}}{4\varepsilon^*\delta\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\delta}}\right) \right] \qquad \text{at} \quad \eta = \eta_0$$

$$(206)$$

It should be noted that this initial N-wave is different from those in the previous cases. The N-wave consists of two expansion wavefronts and a compression region between them. We will show that such N-wave will distort to an "I" wave as it diverges. The method of matched asymptotic expansions is again used here.

(i) Outer Solution

We expand the dependent variable $u_1(\eta, \zeta^*; \varepsilon^*)$ in power series of ε^* and substitute the expansion into Eq.(205) and condition (206), the first term in the expansion satisfies the equation

$$-v_0 \frac{\partial v_0}{\partial \zeta^*} + \frac{\partial v_0}{\partial \eta} + \frac{v}{\eta} = 0 \qquad (207)$$

with the initial condition at $\eta = \eta_0$

$$v_0(\eta, \zeta^*) = \begin{cases} \dfrac{\zeta^*}{\eta_0 \frac{\delta}{\varepsilon\Gamma}(1 - \frac{\varepsilon\Gamma}{\delta}\ln\delta)} & |\zeta^*| \le \frac{\delta}{\varepsilon\Gamma}\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\delta} \\[4mm] 0 & |\zeta^*| > \frac{\delta}{\varepsilon\Gamma}\sqrt{1 - \frac{\varepsilon\Gamma}{\delta}\ln\delta} \end{cases}$$

The solution is

$$\mathcal{V}_0 = \frac{-\xi^*}{\eta\left[\ln\frac{\eta}{\eta_0} - \left(\frac{\delta}{\varepsilon\Gamma} - \sqrt{\ln\frac{1}{\delta}}\right)\right]} \qquad \eta \gtrsim \eta_0 \tag{208}$$

The solution gives the slope of the linear profile (compression wave) between the two expansion wavefronts. It is interesting to note that the slope increases as η increases. The factor in the denominator $\left[\ln\frac{\eta}{\eta_0} - \left(\frac{\delta}{\varepsilon\Gamma} + \sqrt{\ln\frac{1}{\delta}}\right)\right]$ plays an important role in determining pulse-form. As the pulse diverges to the distance $\eta = \eta_0 \exp\left(\frac{\delta}{\varepsilon\Gamma} + \sqrt{\ln\frac{1}{\delta}}\right)$, the slope becomes infinite and the shock wave developes at this distance. We will study the propagation of the diverging pulse in the range $\eta_0 \lesssim \eta < \eta_0 \exp\left(\frac{\delta}{\varepsilon\Gamma} + \sqrt{\ln\frac{1}{\delta}}\right)$ and in the range $\eta \gtrsim \eta_0 \exp\left(\frac{\delta}{\varepsilon\Gamma} + \sqrt{\ln\frac{1}{\delta}}\right)$ separately.

In the range $\eta_0 \lesssim \eta < \eta_0 \exp\left(\frac{\delta}{\varepsilon\Gamma} + \sqrt{\ln\frac{1}{\delta}}\right)$:

Integrating Eq. (207) and using the conditions $\mathcal{V}_0$, $\frac{\partial \mathcal{V}_0}{\partial \xi^*} \to 0$ as $\xi^* \to \pm\infty$, we obtain the relation

$$\eta \, \mathcal{V}_{0m} \, \xi_m^* = \frac{\delta}{\varepsilon\Gamma} \tag{209}$$

It should be noted that the amplitudes of the disturbances at the front, the central, and the rear points of the pulse are zero; these points propagate with constant speed a_0. Therefore, the relative positions among them should remain constant, i.e., $\xi_m^* = $ constant, if no multiple values of the gas variables occur. The slope of the linear profile and the amplitude change as the pulse diverges. From (209) we find

$$\mathcal{V}_{0m} = \frac{\frac{\delta}{\varepsilon\Gamma}}{\eta \, \xi_m^*} \propto \frac{1}{\eta}$$

Hence, in this range of η the maximum amplitude of the diverging pulse decays only due to the geometrical divergence. The pulse-form is shown below:
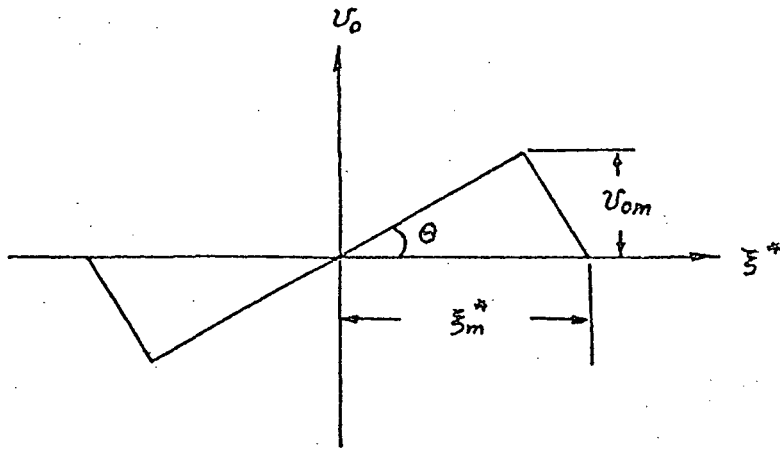
In the range $\eta \geq \eta_o e^{\frac{\delta}{\varepsilon \Gamma} + \sqrt{\ln \frac{1}{\delta}}}$:

As the diverging pulse arrives at the distance $\eta = \eta_o e^{\frac{\delta}{\varepsilon \Gamma} + \sqrt{\ln \frac{1}{\delta}}}$, the slope of the linear profile becomes infinite $(\theta = 90^\circ)$. If $\eta > \eta_o e^{\frac{\delta}{\varepsilon \Gamma} + \sqrt{\ln \frac{1}{\delta}}}$, the slope becomes negative and the gas variables become multiple values, which are physically impossible; a shock wave must occur at $\xi^* = 0$. The pulse-form in this range we will call an "I" wave which is shown below:
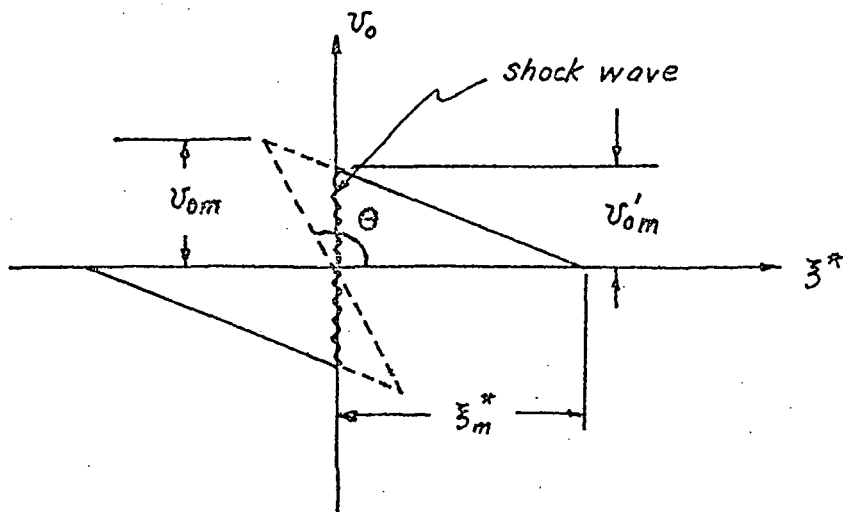
The presence of the shock wave causes energy dissipation, therefore, it will accelerate the amplitude decay. The shock strength $\mathcal{V}'_{om}$ can be calculated as follows:

$$\frac{\mathcal{V}'_{om}}{\mathcal{V}_{om}} = \frac{\xi^*}{\xi^*_m + |\mathcal{V}_{om}\cot\theta|} = \frac{\xi^*_m}{\xi^*_m + \left|\mathcal{V}_{om}\eta\left[\ln\frac{\eta}{\eta_o} - \left(\frac{\delta}{\varepsilon r} + \sqrt{\ln\frac{1}{\delta}}\right)\right]\right|}$$

As η becomes large, we have

$$\mathcal{V}'_{om} \sim \frac{1}{\eta\ln\eta}$$

Thus, the maximum pulse amplitude (shock strength) decays more rapidly than an N-wave because of the initial pulse-form.

To study the shock structure we introduce the following coordinates and expansion:

$$\bar{\xi} = \frac{\xi^*}{\varepsilon^*}$$

$$\eta = \eta$$

$$u_1(\eta, \xi^*; \varepsilon^*) = \mathcal{V}_B(\eta, \xi^*) + \varepsilon^* \mathcal{V}_{B1}(\eta, \xi^*) + \cdots$$

Substituting the above relations into Eq.(205), we find that $\mathcal{V}_B$ satisfies

$$-\mathcal{V}_B \frac{\partial \mathcal{V}_B}{\partial \bar{\xi}} = \frac{\partial^2 \mathcal{V}_B}{\partial \bar{\xi}^2}$$

The boundary conditions are

$$\mathcal{V}_B \longrightarrow \pm \mathcal{V}'_{om} \qquad \text{as} \qquad \bar{\xi} \longrightarrow \pm\infty$$

$$\mathcal{V}_B = 0 \qquad \text{at} \qquad \bar{\xi} = 0$$

The solution is

$$\mathcal{V}_\beta = \mathcal{V}_{om}' \tanh\left(\frac{\mathcal{U}_{m}' \, \mathfrak{z}^*}{2\,\varepsilon^*}\right)$$

This is again a Taylor's shock structure, in which $\mathcal{V}_{om}'$ decreases as η increases ; thus the shock thickness grows as the pulse diverges. However, this solution is not valid for very large η when the shock thickness is comparable to the wavelength.

VI. CONCLUSION

The focusing of spherical and cylindrical pulses in a viscous and thermally conducting medium has been investigated. This problem is governed by three parameters ε , δ , and R_e^{-1} which characterize the initial strength and position of disturbances and nature of the medium in which the pulse propagates. The fundamental assumption here is that all three parameter are small. It is found that a simple and efficient way to study the focusing pulse problem is to divide the region of interest into three regions, namely, the converging region, the interaction region, and the diverging region.

We consider the cylindrical and spherical pulses of infinitesmal amplitude in a viscous and heat-conducting medium ($\frac{\varepsilon}{\delta} \ll 1$, $\delta R_e = 0(1)$, and $\delta \ll 1$); the approximate equation for the converging region is linear and contains the viscous and heat-conduction term. The solution of this equation shows that the amplitude of the pulse decreases initially because the diffusion effects dominate, then increases rapidly due to the strong focusing effect as $r \to 0$. Any initial discontinuity contained in the pulse profile is diffused to a transition zone whose width becomes same order as the pulse-length as the pulse approaches the interaction region, e.g., an initial N-wave with jump discontinuities at front and rear fronts becomes a very smooth profile as it approaches the interaction. The interaction region is found to be governed to first order by the linear wave equation with radial symmetry. The solution of the linear wave equation, which matches the solution of converging region and

satisfies the boundary condition at axis or center of symmetry, shows that the amplitude of the focusing pulse remains small in the whole interaction region including the axis or center of symmetry. The velocity perturbation is found to be linearly proportional to the radial distance from the origin and is zero at the origin; the pressure and density perturbations are independent of the radial distance in the vicinity of the origin. The peculiar phenomenon of 180-degree phase shift after the reflection at the center of symmetry is found in the spherical case, but no phase shift occurs in the cylindrical case.

We also consider converging cylindrical and spherical pulses of finite amplitude in a medium of very small viscosity and heat conductivity. The approximate equation for the converging region is a nonlinear second-order partial differential equation with a small parameter $\varepsilon^* = \dfrac{\delta}{\varepsilon^2 R_e \Gamma}$. The composite solution (first term of a composite series expansion in terms of ε^*) of the approximate equation describes a converging pulse with changing pulse-form, e.g., a sinusoidal wave with compressions $(p > 0)$ followed by rarefactions $(p < 0)$ distorts to an N-wave. The flow fields inside the front and rear shocks are locally discribed by Taylor's shock structure. The shock thickness diminishes as it converges in the converging region despite the presence of viscosity and heat conduction. The flow field inside the expansion zone between the front and rear shocks of the N-wave is found to be inviscid in the first-order approximation. As the N-wave enters the interaction region, it is shown that no small-perturbation solution for the shocks in the interaction region exists under the conditions $\dfrac{\varepsilon}{\delta} = O(1)$, $\delta R_e \gg 1$. However, the flow field

between the front and rear shocks satisfies to first order the linear wave equation. This approximation is shown to be applicable only if the parameters ε , δ , R_e satisfy certain relations which insure that the amount of the energy dissipated through the shock front of the N-wave is negligible in comparison with the total energy of the N-wave.

The governing equation for the diverging pulse is shown to be identical to that of converging pulse and can be treated by the same method. The solution for the diverging pulse shows the decay of amplitude due to geometrical, nonlinear, and diffusion effects. It also shows the increase of pulse-length due to the nonlinear distortion, viscous and heat-conduction effects.

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